

INSTITIÚID TEICNEOLAÍOCHTA CHEATHARLACH

INSTITUTE OF TECHNOLOGY CARLOW

APPLIED MATHEMATICS - INTRODUCTION

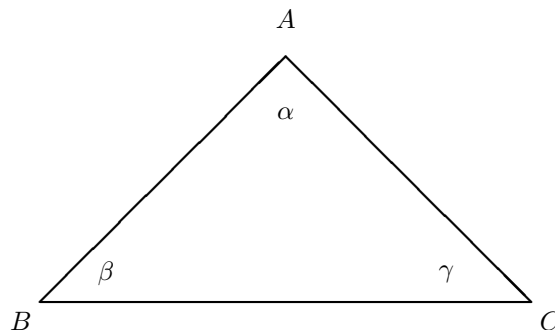
1 Introduction

Our objective is to provide students with the mathematics required to model the real world as applied in computer graphics. Linear Algebra is a fundamental topic of mathematics for students of computer science. An extensive subject, it covers topics such as vectors and matrices, systems of linear equations, determinants, eigenvalue problems and other topics. Linear Algebra has important geometric applications and hence is central to subjects like computer graphics.

To motivate this subject consider the following problem:

Determine the area of a triangle ABC whose vertices are the points

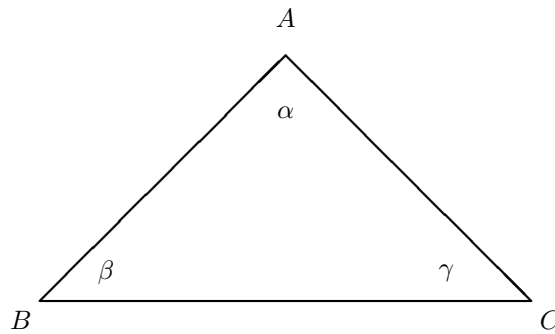
$$A(5, 7) , B(2, 2) , C(8, 3)$$



We have a choice in how to present a solution to this problem. This problem may be solved using results from basic coordinate geometry and trigonometry. The problem may also be solved using the algebra

of vectors in 2–dimensions and basic trigonometry. The results used to solve this problem can be easily extended to 3–dimensions to determine the area of a triangle ABC whose vertices are the points

$$A(1, -1, 0) , B(2, 1, -1) , C(-1, 1, 2)$$



William Rowan Hamilton (1805-1865) was considered by some of his contemporaries to approach Issac Newton in intellectual power. He was born on August 4th, 1805 in Dublin and possibly because of his father's financial circumstances had become strained, William was sent at the age of three to live with his uncle, the Reverend James Hamilton, head of a diocesan school in the village of Trim, Co. Meath. His uncle was a classical scholar who had graduated from Trinity College, Dublin. William had little contact with his parents during childhood. He began his education as soon as he arrived in Trim, quickly revealing himself to be a child prodigy. His uncle was aiming to get him into Trinity College in due course, and gave him every encouragement. Languages came easily to him and by the age of ten he was said to know not only the classical languages of Latin, Greek and Hebrew but also some modern European languages.

His serious mathematical education began at the age of thirteen with the study of Euclid, and went on to read the works of Newton and Laplace.



Figure 1: *William Rowan Hamilton (1805–1865)*

In 1824, at nineteen, Hamilton entered Trinity College and that same year published a note correcting a minor error in Laplace's work. During his four years at Trinity College he was awarded two separate 'optimes' for his performances: so rare was the honour that no optimes at all had been awarded for twenty years. In 1827 the chair of astronomy at Trinity became vacant and Hamilton was invited to apply. In spite of his age and lack of experience Hamilton was appointed Andrews professor of astronomy, and consequently became Astronomer Royal of Ireland and Director of the Observatory at

Dunsink, although he had yet to take his degree. Hamilton retained these latter positions throughout his life, and made the observatory his home. After a few years he gave up his work as a practical astronomer to concentrate on mathematics. During the 1830s he led an extremely active life. These were his most productive years. He made important contributions to the understanding of dynamics and optics, invented quaternions and, in graph theory, developed what he called ‘icosian calculus’.

It was the connection between the algebra of complex numbers and rotations in the plane that intrigued Hamilton for years. His problem was to try to discover an extension to the complex number system that could be used to describe rotations in 3–dimensions, using triples of real numbers instead of pairs. This he was unable to accomplish, for what later turned out to be a very good reason. On 16th October, 1843, Hamilton while walking with his wife along the Royal canal in Dublin, that he had a flash of inspiration when crossing Broome Bridge. In that instant he saw that triples were not enough, but that four-tuples were required. He saw that he needed not just the complex number component i , but rather three such components i , j and k satisfying the relationship

$$i^2 = j^2 = k^2 = ijk = -1$$

So struck was he by this discovery that supposedly he stopped and carved this equation into the stone of Broome Bridge. It was at that moment quaternions were discovered. The plaque to commemorate his discovery was erected on Broome Bridge and has the inscription

Here as he walked by on 16th October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication

$$i^2 = j^2 = k^2 = ijk = -1$$

& cut it on a stone of this bridge.

From 1843 the concept of a quaternion was simply another abstract mathematical idea until the advent of computer graphics almost 100 years later. They are now used in most computer games and graphics programs where objects pass through multiple rotations as part of the animation of the objects on the screen. This topic will attempt to show how complex numbers can be used to rotate



Figure 2: *Plaque (Broome Bridge), Royal Canal, Dublin.*

in 2–dimensions and the extension of complex numbers to quaternions will allow for rotations in 3–dimensions. In doing so we develop the algebra of both complex numbers and quaternions. In defining a complex number we will introduce a new number i such that $i^2 = -1$, i.e. $i = \sqrt{-1}$. This will give rise to further numbers of the form $x + iy$.

Hence, our approach will be to begin with some basic trigonometry. We will define fundamental objects such as a vector, a matrix and a quaternion. The basic algebra of these objects will allow us consider transformations and rotations in 2 and 3–dimensions. Some geometry of Lines and Planes in 3–dimensions will be important here. Where possible some segments of $C++$ are included to support the idea and to connect with the practical associated with the topic.