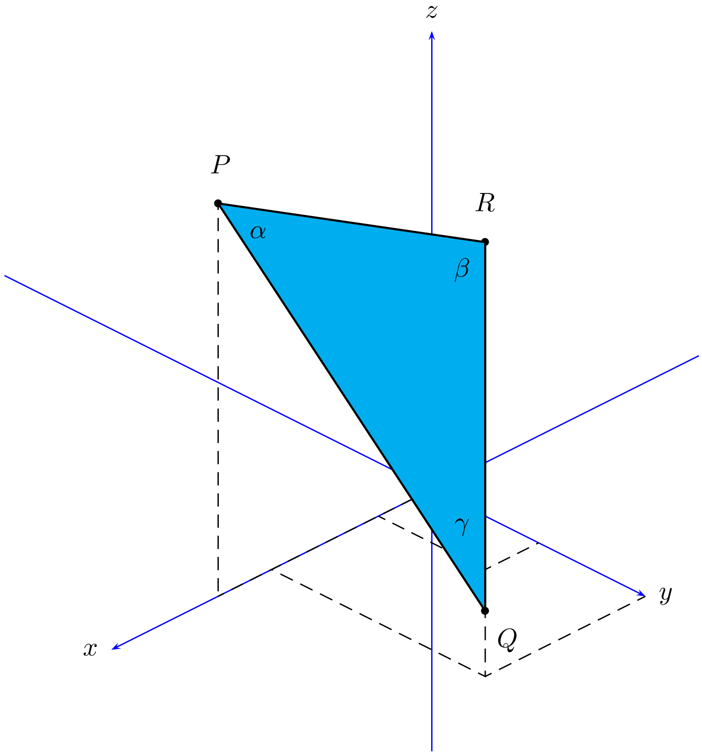
INSTITIÚID TEICNEOLAÍOCHTA CHEATHARLACH

INSTITUTE OF TECHNOLOGY CARLOW

TRIGNOMETRY AND VECTORS

Contents


z
P
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1 Basic Trigonometry

These definitions are required in the design of simple graphics programs and in all computer games.

1.1 Angular Measure

The most common system is *degree measure* in which the complete circle is divided into 360 degrees. For more accurate measurement each degree is divided into 60 minutes and each minute is divided into 60 seconds. So, for example

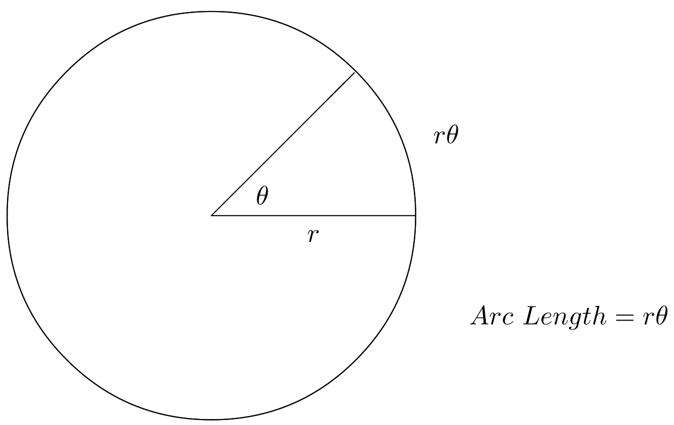
$$25^o30^{'} = 25 \cdot 5^o$$

$$38^o15^{'} = 38 \cdot 25^o$$

$$90^o15^{'}25^{''} = 90 \cdot 2569444^o$$

A mathematically more natural unit of degree measure is the *radian*.

Definition 1 *One radian is the angle at the centre of a circle subtended by an arc whose length is equal to the radius.*



From the diagram and the above definition

$$\frac{r\theta}{r} = \theta = 1 \ radian$$

The length of the circumference of a circle is given as $2\pi r$. Hence

$$\frac{2\pi r}{r} = 2\pi \ radians$$

defines the number of radians in the complete circle. This gives us a relationship between the angle measure of degrees and radians.

$$\begin{aligned} 360^o &= 2\pi \; radians \\ 180^o &= \pi \; radians \quad ** \end{aligned}$$

Exercise Convert each of the following angles in degrees to radians:

$$90^o \qquad 30^o \qquad 270^o \qquad 15 \cdot 382^o$$

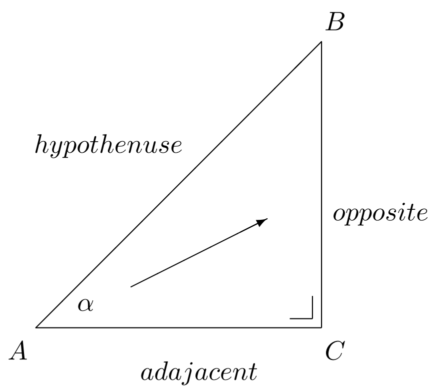
Exercise Convert each of the following angles in radians to degrees:

$$\frac{\pi}{2} \qquad \frac{\pi}{4} \qquad \frac{5\pi}{6} \qquad 1 \cdot 4\pi \qquad 1 \; radian \qquad 2 \cdot 9 \; radians$$

Note All computer languages use only *radian measure*. You as a programmer will have to convert from degrees to radians before any calculations in your programme.

1.2 Basic Trigonometrical Functions

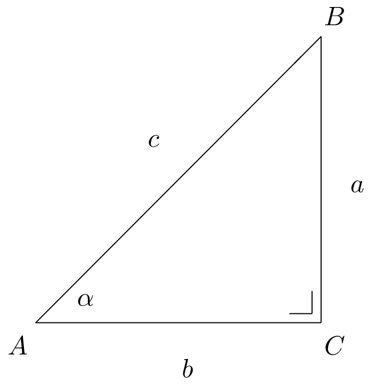
We can define the three trigonometric functions $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ by use of a right-angled triangle. Consider the triangle ABC



$$\begin{aligned} \sin\alpha &= \frac{opposite}{hypothenuse} \\ \cos\alpha &= \frac{adjacent}{hypothenuse} \\ \tan\alpha &= \frac{opposite}{adjacent} = \frac{\sin\alpha}{\cos\alpha} \end{aligned}$$

Theorem 1 *(Pythagoras' theorem) In a right-angled triangle , the sum of the squares of the lengths of the sides containing the right angle is equal to the square of the hypothenuse; i.e.*

$$a^2 + b^2 = c^2$$



Three positive integers a, b and c such that $a^2 + b^2 = c^2$ are called *Pythagorean triples*. For example $(3, 4, 5)$, $(5, 12, 13)$, $(6, 8, 10)$, $(8, 15, 17)$, $(9, 12, 15)$ are all solutions of the equation

$$a^2 + b^2 = c^2$$

Remark In the early 1600's, Pierre de Fermat (1601–1665), a French lawyer and mathematician posed the following question – if the power of 2 in the above equation was replaced by 3 could there be found three non-zero integers a, b and c that satisfy the equation $a^3 + b^3 = c^3$? The same question could be asked if the power was increased to 4 then to 5 and down to any positive integer n.

$$\begin{aligned} a^3 + b^3 &= c^3 \\ a^4 + b^4 &= c^4 \\ &\dots \\ &\dots \\ a^n + b^n &\neq c^n \end{aligned}$$

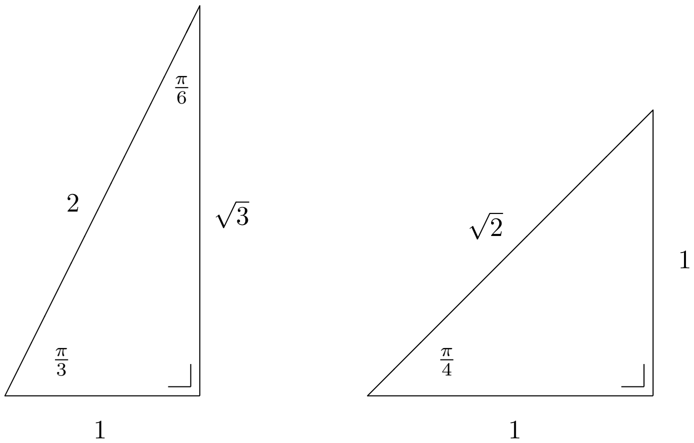
Fermat stated that the no matter how hard you try you will never find integer solutions to these equations. This famous statement become known as Fermat's 'Last' Theorem, which was not solved until 1994 by British-American mathematician Andrew Wiles. Wiles devoted seven years of his life to proving the famous theorem, which may have generated more attempts at proofs than any other theorem.


Pierre de Fermat (1601–1665)

Fermat's 'Last' Theorem states that $a^n + b^n = c^n$ has no non-zero integer solutions for a, b and c when $n > 2$. Fermat stated his theorem in 1637 when he wrote that "I have a truly marvelous" proof of this proposition which this margin is too narrow to contain". Today, however, we believe that Fermat had no such proof.

Remark Recall that there are simple exact expressions for the sine and cosine of the angles

$$\frac{\pi}{6} \quad , \quad \frac{\pi}{4} \quad , \quad \frac{\pi}{3}$$



It follows from these right-angled triangles that

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \quad , \quad \sin\frac{\pi}{6} = \frac{1}{2} \quad , \quad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$\cos\frac{\pi}{3} = \frac{1}{2} \quad , \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad , \quad \tan\frac{\pi}{3} = \sqrt{3}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad , \quad \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad , \quad \tan\frac{\pi}{4} = 1$$

Note

i The *sine*, *cosine* and *tangent* of an angle may be calculated on a scientific calculator. If the angle measure is in *degrees* your calculator must be in degree mode (**deg**). Most calculators will default to this mode in order to proceed. If the angle measure is in *radians* your calculator must be in radian mode (**rad**) in order to proceed.

ii All scientific calculators *inverse sine*, *inverse cosine* and *inverse tangent* functions. These trigonometrical functions are denoted as

$$\sin^{-1} \quad , \quad \cos^{-1} \quad , \quad \tan^{-1}$$

Exercise Calculate each of the following, writing each answer accurate to four places of decimals:

$$\sin 42\cdot 38^o \qquad \sin\frac{2\pi}{3} \qquad \sin 50^o \qquad \sin(1\cdot 6)\ radians$$

[**Solution**: *0·6740* , *0·8660* , *0·7660* , *0·9996*].

Exercise Find the value of A, accurate to 3 decimal places in each of the following equations:

$$2\cos 3B = A \quad , \quad when\ B = 10^o$$

$$2A\tan 5B = 0\cdot 5 \quad , \quad when\ B = 1\ radian$$

$$2\sin(3B + \frac{\pi}{3}) = 5A \quad , \quad when\ B = \frac{\pi}{6}$$

[**Solution**: *1·732* , *-0·074* , *0·200*].

To illustrate the use of an *inverse trigonometrical function* consider the following example:

Example To determine the angle α when

$$2\sin 3\alpha = 0\cdot 5\cos\frac{\pi}{4}$$

we have

$$\begin{aligned}
2\sin 3\alpha &= 0\cdot 5\cos\frac{\pi}{4} \\
2\sin 3\alpha &= 0\cdot 35355 \\
\sin 3\alpha &= 0\cdot 17678 \\
3\alpha &= \sin^{-1} 0\cdot 17678 \\
3\alpha &= 0\cdot 17777\ rad \quad (10\cdot 182^o) \\
\therefore \quad \alpha &= 0\cdot 05924\ rad \quad (3\cdot 394^o)
\end{aligned}$$

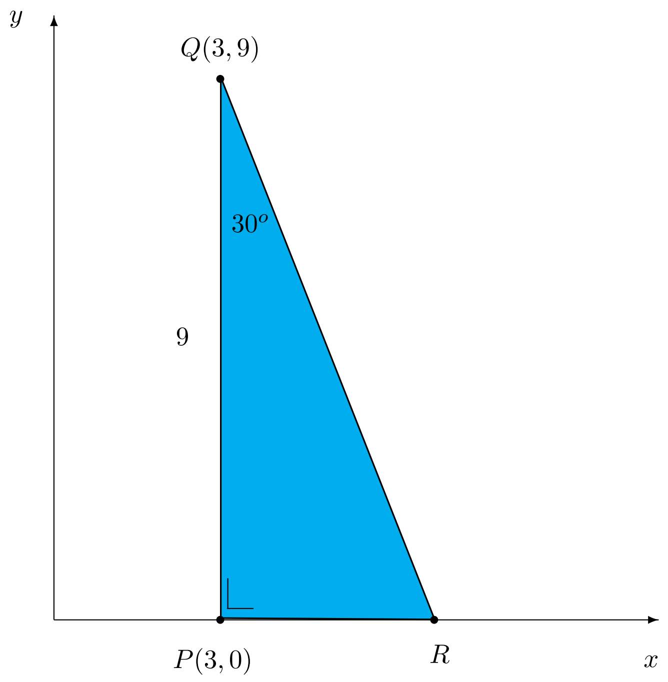
Exercise Find the angle α, giving your answer in degrees accurate to 2 decimal places, in each of the following equations:

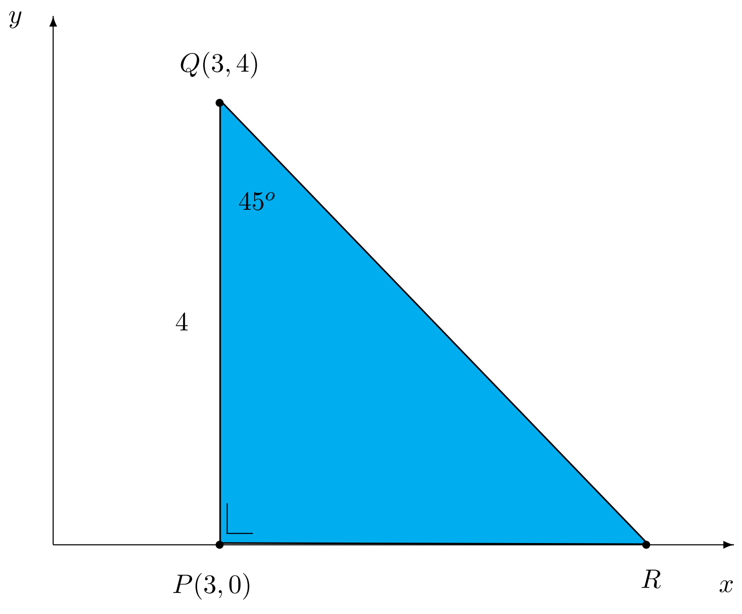
$$2\sin 3\alpha = 1\cdot 4\tan 40^o$$

$$\tan\left(2\alpha - \frac{\pi}{2}\right) = 5\cdot 2$$

[**Solution**: *11·99^o , 84·56^o*].

Exercise In each of the following coordinate diagrams in $\mathbb{R}^2$ the triangle PQR is a right-angled triangle. In each case, using *basic trigonometry*, determine the coordinates of the point R.





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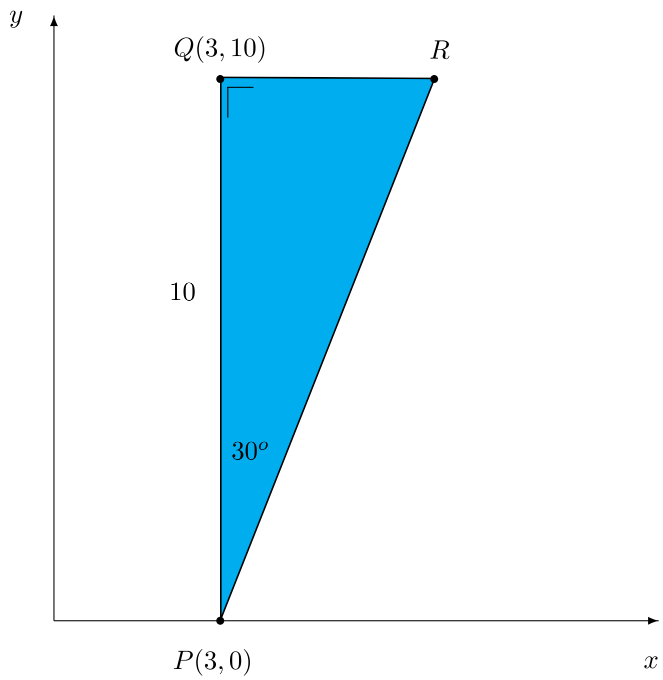
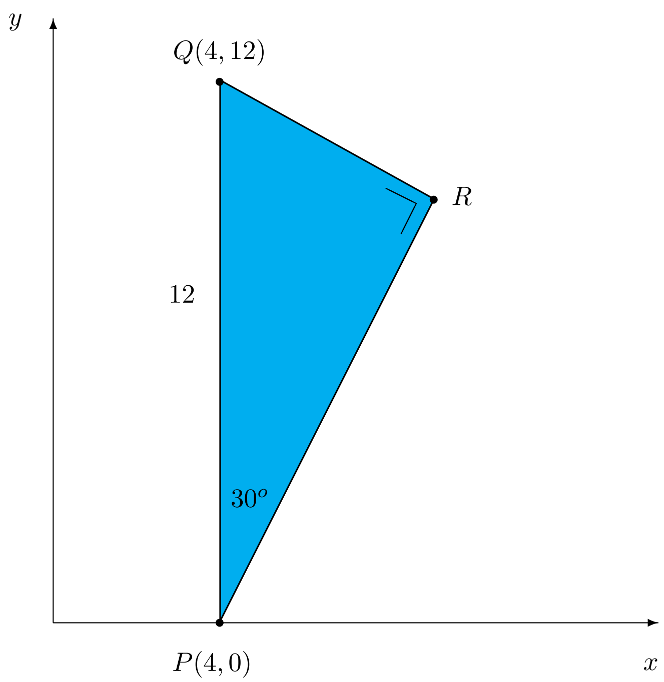




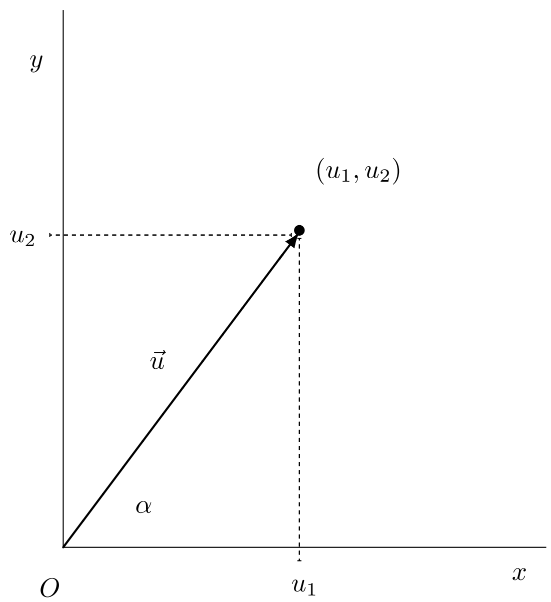
2 Vectors in Two Dimensions

A vector is an object which has *magnitude* and *direction.* Many physical quantities, such as velocity, acceleration, force, electric field and magnetic field are examples of vector quantities. Displacement between points may also be represented using vectors. We study some relationship between algebra and geometry. We shall first study some algebra which is motivated by geometric considerations. We then use the algebra later to better understand some problems in geometry. This mathematics will form the basis of the study of computer graphics. Vectors are central to the design of any two-dimensional or three-dimensional computer game. They are used to represent points in space, like corners of a door or window or the location of any object in a scene. They are also used to describe a direction, for example the orientation of a camera or the direction in which a gun is pointing.

A vector in two-dimensions $\mathbb{R}^2$ can be described as an ordered pair $\vec{u} = (u_1, u_2)$, where $u_1, u_2 \in \mathbb{R}$.

Definition 2 *Two vectors* $\vec{u} = (u_1, u_2)$ *and* $\vec{v} = (v_1, v_2)$ *in* $\mathbb{R}^2$ *are said to be equal, denoted by* $\vec{u} = \vec{v}$, *if and only if* $u_1 = v_1$ *and* $u_2 = v_2$.

Defining vectors in $\mathbb{R}^2$ as ordered pairs of real numbers enables us to state precisely when two vectors are equal – it also provides us with the easiest way of defining addition and various kinds of multiplication. To describe the position of any point in two-dimensions we may choose two axes x and y which are mutually perpendicular and intersect in a point O called the origin, as shown.



Any point P in two dimensions corresponds the ordered pair (u_1, u_2) of real numbers, where u_1 represents the magnitude of the component vector along x-axis and u_2 represents the magnitude of the component vector along y-axis.

The magnitude of the vector $\vec{u}$, from Pythagoras' theorem, is given as

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2}$$

The direction of the vector $\vec{u}$ is defined by α

$$\alpha = tan^{-1}\left(\frac{u_2}{u_1}\right)$$

2.1 Addition of Vectors

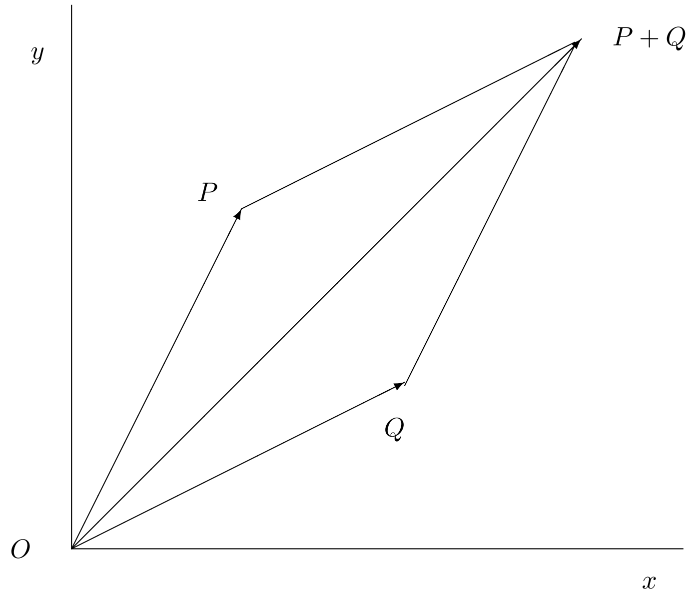
Definition 3 *For any two vectors* $\vec{u} = (u_1, u_2)$ *and* $\vec{v} = (v_1, v_2)$ *in* $\mathbb{R}^2$*, we define their sum to be*

$$\begin{aligned} \vec{u} + \vec{v} &= (u_1, u_2) + (v_1, v_2) \\ &= (u_1 + v_1, u_2 + v_2) \end{aligned}$$

Similarly, we define their difference to be

$$\begin{aligned} \vec{u} - \vec{v} &= (u_1, u_2) - (v_1, v_2) \\ &= (u_1 - v_1, u_2 - v_2) \end{aligned}$$

Addition of vectors may be pictured using the 'parallelogram law'.



Example For the following pair of vectors $\vec{u} = (1, 6)$ and $\vec{v} = (-5, 2)$ in $\mathbb{R}^2$, we can calculate

$$\begin{aligned} \vec{u} + \vec{v} &= (1,6) + (-5,2) \\ &= (-4,8) \\ \vec{u} - \vec{v} &= (1,6) - (-5,2) \\ &= (6,4) \end{aligned}$$

Theorem 2 *(VECTOR ADDITION)*

i *For every* $\vec{u}, \vec{v} \in \mathbb{R}^2$*, we have* $\vec{u} + \vec{v} \in \mathbb{R}^2$.

ii *For every* $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$*, we have* $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.

iii *For every* $\vec{u} \in \mathbb{R}^2$*, we have* $\vec{u} + 0 = \vec{u}$ *where* $0 = (0, 0) \in \mathbb{R}^2$.

iv *For every* $\vec{u} \in \mathbb{R}^2$*, there exists* $\vec{v} \in \mathbb{R}^2$ *such that* $\vec{u} + \vec{v} = 0$.

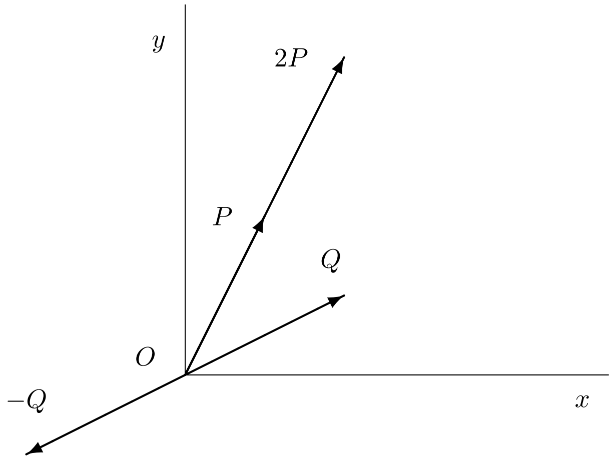
v *For every* $\vec{u}, \vec{v} \in \mathbb{R}^2$*, we have* $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

2.2 Scalar Multiplication of Vectors

Definition 4 *For any vector* $\vec{u} = (u_1, u_2)$ *in* $\mathbb{R}^2$ *and any scalar* $c \in \mathbb{R}$*, we define the scalar multiple to be*

$$c\vec{u} = c(u_1, u_2) = (cu_1, cu_2)$$

Scalar multiplication may be pictured as follows –



Example For the following pair of vectors $\vec{u} = (2,3)$ and $\vec{v} = (-1,5)$ in $\mathbb{R}^2$, we can evaluate

$$\begin{aligned} 2\vec{u} + 4\vec{v} &= 2(2,3) + 4(-1,5) \\ &= (4,6) + (-4,20) \\ &= (0,26) \\ 2\vec{u} - \vec{v} &= 2(2,3) - (-1,5) \\ &= (4,6) - (-1,5) \\ &= (5,1) \end{aligned}$$

Theorem 3 *(SCALAR MULTIPLICATION)*

i For every $c \in \mathbb{R}$ and $\vec{u} \in \mathbb{R}^2$, we have $c\vec{u} \in \mathbb{R}^2$.

ii For every $c \in \mathbb{R}$ and $\vec{u}, \vec{v} \in \mathbb{R}^2$, we have $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$.

iii For every $a, b \in \mathbb{R}$ and $\vec{u} \in \mathbb{R}^2$, we have $(a+b)\vec{u} = a\vec{u} + b\vec{u}$.

iv For every $a, b \in \mathbb{R}$ and $\vec{u} \in \mathbb{R}^2$, we have $(ab)\vec{u} = a(b\vec{u})$.

v For every $\vec{u} \in \mathbb{R}^2$, we have $1\vec{u} = \vec{u}$.

Exercise For the following pair of vectors $\vec{u} = (4,-3)$ and $\vec{v} = (1,7)$ in $\mathbb{R}^2$, evaluate

i $3\vec{u} + 3\vec{v}$,

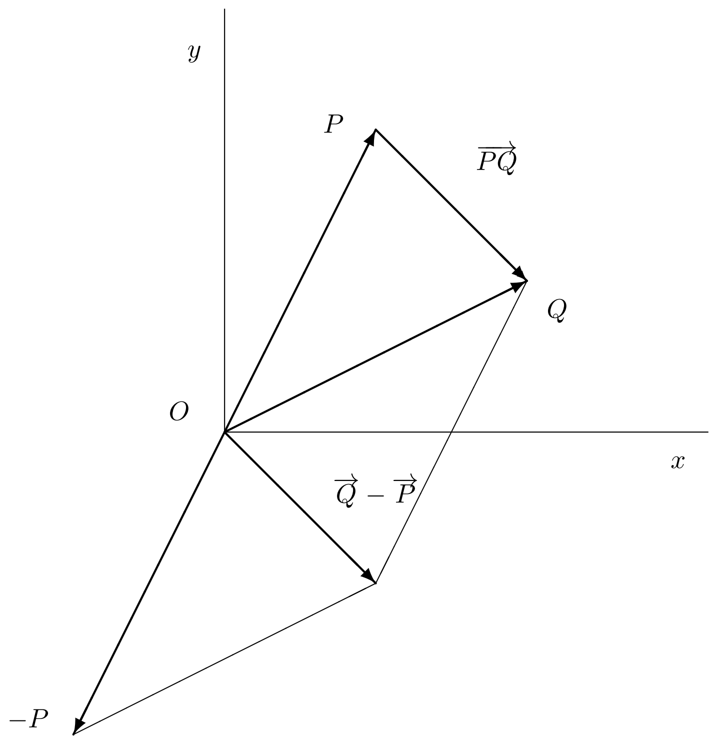
ii $\vec{u} + 2\vec{v}$,

iii $7\vec{u} - 3\vec{v}$.

Remark There is another way in which vectors may be pictured – namely as 'arrows' in two dimensions. The vector (u_1, u_2) can be pictured by an arrow with initial point O and terminal point (u_1, u_2). It is, however, convenient to picture vectors in a more general way. Consider an arrow with the initial point $P = (x_1, y_1)$ and terminal point $Q = (x_2, y_2)$. This arrow is denoted by $\overrightarrow{PQ}$. We define

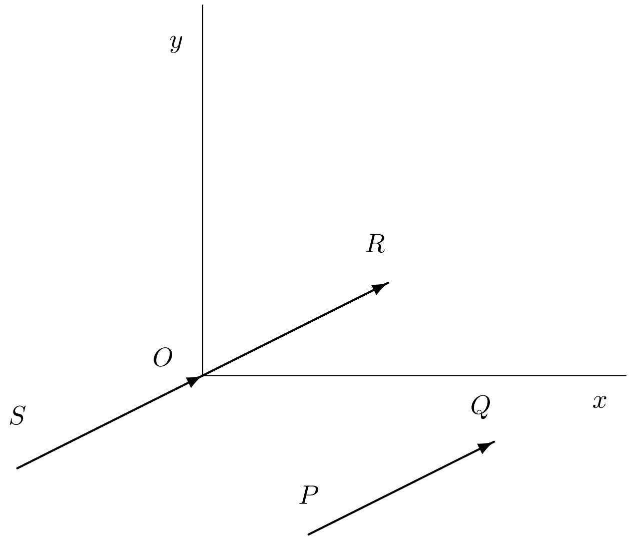
$$\begin{aligned} \overrightarrow{PQ} &= (x_2 - x_1, y_2 - y_1) \\ &= \vec{Q} - \vec{P} \end{aligned}$$

We can picture this as follows – using the parallelogram law



Note that this vector is already represented by the arrow with initial point O and terminal point $(x_2 - x_1, y_2 - y_1)$. In fact, any one vector may be represented by infinitely many arrows. We define two arrows to be *equivalent* whenever their corresponding components are equal – it is the components, and not the individual initial and terminal points, which are used to see if two arrows are equivalent. Since components are determined by the length and direction of an arrow we can state that two arrows are equivalent whenever they have the same length and direction. Since one vector is now represented by any one of infinitely many equivalent arrows, we agree to regard these equivalent arrows as equal. The end result is that we may picture a vector as an arrow which has a given length and lies in a given direction, and may be positioned between any pair of points provided that the points determine the same length and direction.

Example Let $P = (2, -5)$, $Q = (3, -2)$, $R = (1, 3)$ and $S = (-1, -3)$ be four points in two dimensions, as shown.



Then

$$\begin{aligned} \overrightarrow{PQ} &= (1, 3) \\ \overrightarrow{OR} &= (1, 3) \\ \overrightarrow{SO} &= (1, 3) \end{aligned}$$

These arrows represent the same vector, namely, $A = (1, 3)$ and we write

$$\overrightarrow{PQ} = \overrightarrow{OR} = \overrightarrow{SO} = A = (1, 3)$$

Exercise In each case write the vector $\vec{u}$ in terms of components

i $\vec{u}$ is a vector from the point $A(2, -5)$ to the point $B(0, 4)$,

ii $\vec{u}$ is a vector from the point $A(-1, -3)$ to the point $B(5, 2)$,

iii $\vec{u}$ is a vector from the point $A(5, 12)$ to the point $B(-3, -6)$.

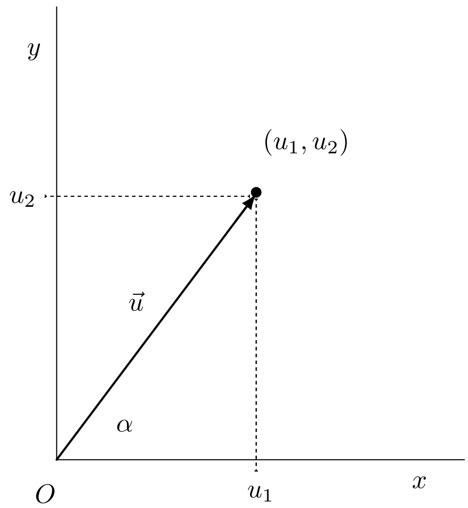
2.3 Magnitude and Direction

Definition 5 *For any vector $\vec{u} = (u_1, u_2)$ in $\mathbb{R}^2$, we define the magnitude of $\vec{u}$ to be the non-negative real number*

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2}$$

The direction of the vector $\vec{u}$ is defined by α

$$\alpha = tan^{-1}\left(\frac{u_2}{u_1}\right)$$



Remark

i Suppose that $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are two points in $\mathbb{R}^2$. To calculate the distance $d(P, Q)$ between the two points, we must first find a vector from P to Q. This is given by $(x_2 - x_1, y_2 - y_1)$. The distance $d(P, Q)$ is then the magnitude of this vector, so that

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Hence, the definition of the magnitude (or norm) of a vector $\vec{u}$ is simply the distance from O to the point (u_1, u_2).

ii A vector of magnitude 1 is called a *unit vector* or *normalized vector*. Any non-zero vector $\vec{u}$ determines a unit vector

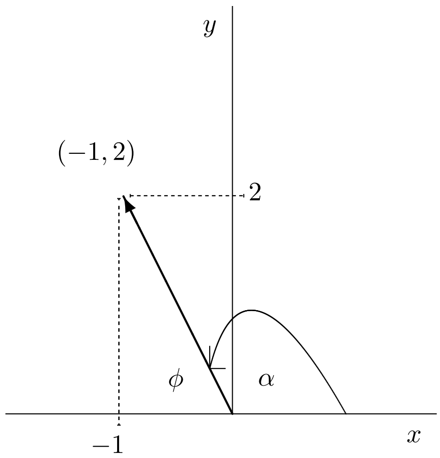
$$\frac{1}{\|\vec{u}\|}\vec{u} = \left(\frac{u_1}{\|\vec{u}\|}, \frac{u_2}{\|\vec{u}\|}\right)$$

Example The vector $\vec{u} = (3, 4)$ has magnitude 5. This vector has direction $\alpha = 53 \cdot 13^o$.

Example The vector $\vec{u} = (2, 5)$ has magnitude $5 \cdot 385$. This vector has direction $\alpha = 68 \cdot 199^o$.

Example The vector $\vec{u} = (1, 2)$ has magnitude $2 \cdot 236$. This vector has direction $\alpha = 63 \cdot 434^o$.

Example The vector $\vec{u} = (-1, 2)$ has magnitude $2 \cdot 236$.



From the diagram

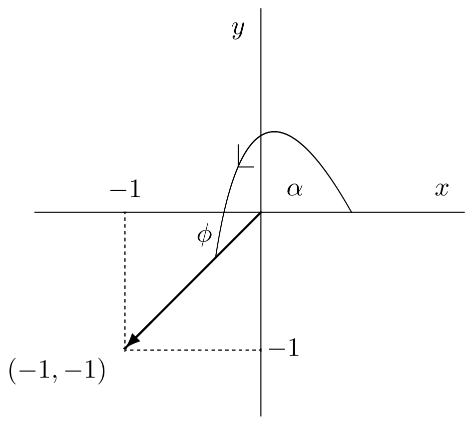
$$\phi = tan^{-1}\left(\frac{2}{1}\right) = 63 \cdot 43^o$$

Hence $\alpha = 180^o - 63 \cdot 43^o = 116 \cdot 57^o$.

Note The angle of direction α is the always quoted relative to the **positive x–axis**.

Example The vector $\vec{u} = (-2, 3)$ has magnitude $3 \cdot 606$. This vector has direction $\alpha = 123 \cdot 69^o$.

Example The vector $\vec{u} = (-1, -1)$ has magnitude $1 \cdot 414$.



From the diagram

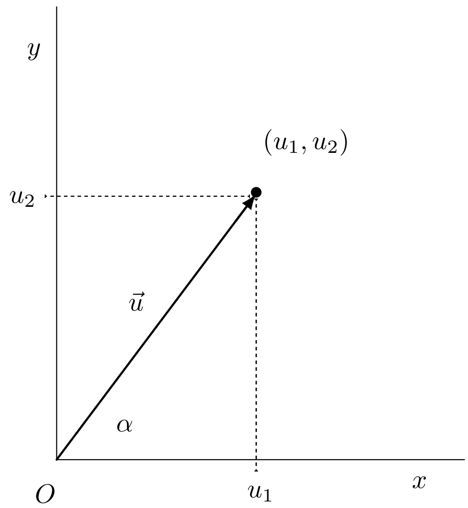
$$\phi = tan^{-1}\left(\frac{1}{1}\right) = 45^o$$

Hence $\alpha = 180^o + 45^o = 225^o$.

Example The vector $\vec{u} = (-4, -5)$ has magnitude $6 \cdot 403$. This vector has direction $\alpha = 231 \cdot 34^o$.

Example The vector $\vec{u} = (-9, -12)$ has magnitude 15. This vector has direction $\alpha = 233 \cdot 13^o$.

Remark We may be required to determine the components u_1 and u_2 of the vector $\vec{u} = (u_1, u_2)$ when presented with the magnitude and direction of $\vec{u}$ only. If, for example, the magnitude and direction of two distinct vectors are presented – converting each vector to components will allow for simpler addition (subtraction or scalar multiplication) of the vectors.



For $\vec{u} = (u_1, u_2)$ we can write,

$$\begin{aligned} u_1 &= \|\vec{u}\| \cos\alpha \\ u_2 &= \|\vec{u}\| \sin\alpha \end{aligned}$$

Example The vector $\vec{u}$ with magnitude 5 and direction $\alpha = 40^o$ with positive x–axis has components

$$\begin{aligned} u_1 &= 5\cos 40^o \\ u_2 &= 5\sin 40^o \end{aligned}$$

Hence $\vec{u} = (3 \cdot 83\ ,\ 3 \cdot 21)$.

Example The vector $\vec{u}$ with magnitude 200 and direction $\alpha = 210^o$ with positive x–axis has components

$$\begin{aligned} u_1 &= 200\cos 210^o \\ u_2 &= 200\sin 210^o \end{aligned}$$

Hence $\vec{u} = (-173 \cdot 21\ ,\ -100)$.

Example The vector $\vec{u}$ has magnitude 10 and direction $\alpha = 45^o$ with positive x–axis. The vector $\vec{v}$ has magnitude 15 and direction $\alpha = 205^o$ with positive x–axis.

Determine the magnitude and direction of each of the following vectors:

i $\vec{u} + \vec{v}$

ii $\vec{u} - 2\vec{v}$

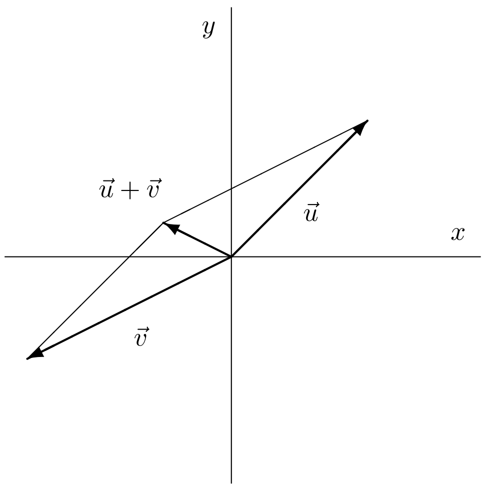
iii $2\vec{u} - 3\vec{v}$

Solution: Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$. Now

$$\begin{aligned} u_1 &= 10\cos 45^o = 7 \cdot 07 \\ u_2 &= 10\sin 45^o = 7 \cdot 07 \\ v_1 &= 15\cos 205^o = -13 \cdot 59 \\ v_2 &= 15\sin 205^o = -6 \cdot 34 \end{aligned}$$

Hence $\vec{u} = (7 \cdot 07, 7 \cdot 07)$ and $\vec{v} = (-13 \cdot 59, -6 \cdot 34)$.

i



$$\begin{aligned}\vec{u}+\vec{v} &= (7\cdot 07, 7\cdot 07)+(-13\cdot 59, -6\cdot 34)\\ &= (7\cdot 07-13\cdot 59, 7\cdot 07-6\cdot 34)\\ &= (-6\cdot 52, 0\cdot 73)\end{aligned}$$

The vector $\vec{u}+\vec{v}$ has magnitude $6\cdot 56$. This vector has direction $\alpha = 96\cdot 39^o$.

ii

$$\begin{aligned}\vec{u}-2\vec{v} &= (7\cdot 07, 7\cdot 07)-2(-13\cdot 59, -6\cdot 34)\\ &= (7\cdot 07, 7\cdot 07)-(-27\cdot 18, -12\cdot 68)\\ &= (7\cdot 07+27\cdot 18, 7\cdot 07+12\cdot 68)\\ &= (34\cdot 25, 19\cdot 75)\end{aligned}$$

The vector $\vec{u}-2\vec{v}$ has magnitude $39\cdot 54$. This vector has direction $\alpha = 29\cdot 97^o$.

iii

$$\begin{aligned}2\vec{u}-3\vec{v} &= 2(7\cdot 07, 7\cdot 07)-3(-13\cdot 59, -6\cdot 34)\\ &= (14\cdot 14, 14\cdot 14)-(-40\cdot 77, -19\cdot 02)\\ &= (14\cdot 14+40\cdot 77, 14\cdot 14+19\cdot 02)\\ &= (54\cdot 91, 33\cdot 16)\end{aligned}$$

The vector $2\vec{u}-3\vec{v}$ has magnitude $64\cdot 15$. This vector has direction $\alpha = 31\cdot 13^o$.

Remark For computer games and graphics programming the component representation of a vector with round (or square) brackets is used. This notation will be used in all modern computer programming languages. In *C#*, for example, the **DrawLine()** method draws a line from one vector point $(x1, y1)$ to a second vector point $(x2, y2)$ on the graphics form:

$$\mathbf{g.DrawLine(p, x1, y1, x2, y2); //Drawline\ method}$$

The following lines of code from *C#* will draw a line from the point (100, 150) to (300, 400):

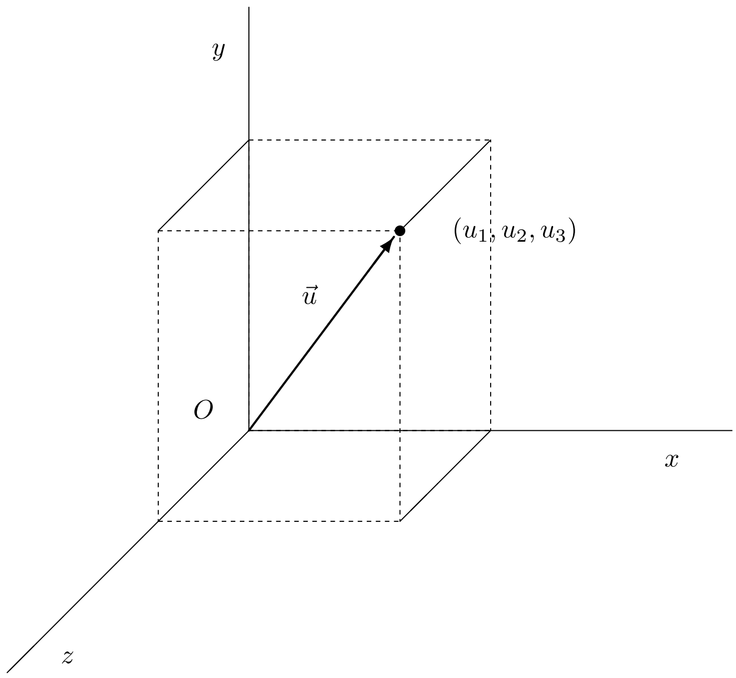
```
privatevoidForm1Paint(object sender, PaintEventArgs e)
{
Graphics g = e.Graphics; //The graphics class
Pen p = newPen(Color.Red); //The Pen class
g.DrawLine(p, 100, 150, 300, 400); //Drawline method
}
```

3 Vectors in Three Dimensions

A vector in three-dimensions $\mathbb{R}^3$ can be described as an ordered triple $\vec{u} = (u_1, u_2, u_3)$, where $u_1, u_2, u_3 \in \mathbb{R}$.

Definition 6 *Two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ are said to be equal, denoted by $\vec{u} = \vec{v}$, if and only if, $u_1 = v_1$, $u_2 = v_2$ and $u_3 = v_3$.*

Defining vectors in $\mathbb{R}^3$ as ordered triples of real numbers enables us to state precisely when two vectors are equal – it also provides us with the easiest way of defining addition and various kinds of multiplication, as we will show later. To describe the position of any point in space, we may choose three axes x, y and z which are mutually perpendicular and intersect in a point O called the origin, as shown.



Any point in space corresponds to the ordered triple (u_1, u_2, u_3) of real numbers, where u_1 represents the magnitude of the component vector along x-axis, u_2 represents the magnitude of the component vector along y-axis and u_3 represents the magnitude of the component vector along the z-axis.

The magnitude of the vector $\vec{u}$, from Pythagoras' theorem, is given as

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

The direction of the vector in three-dimensions $\mathbb{R}^3$ in is defined by three angles θ_x, θ_y and θ_z the vector makes with the x-axis, y-axis and z-axis respectively. For a vector $\vec{u} = (u_1, u_2, u_3)$ in $\mathbb{R}^3$

$$\begin{aligned} u_1 &= \|\vec{u}\| \cos\theta_x \\ u_2 &= \|\vec{u}\| \cos\theta_y \\ u_3 &= \|\vec{u}\| \cos\theta_z \end{aligned}$$

3.1 Addition of Vectors

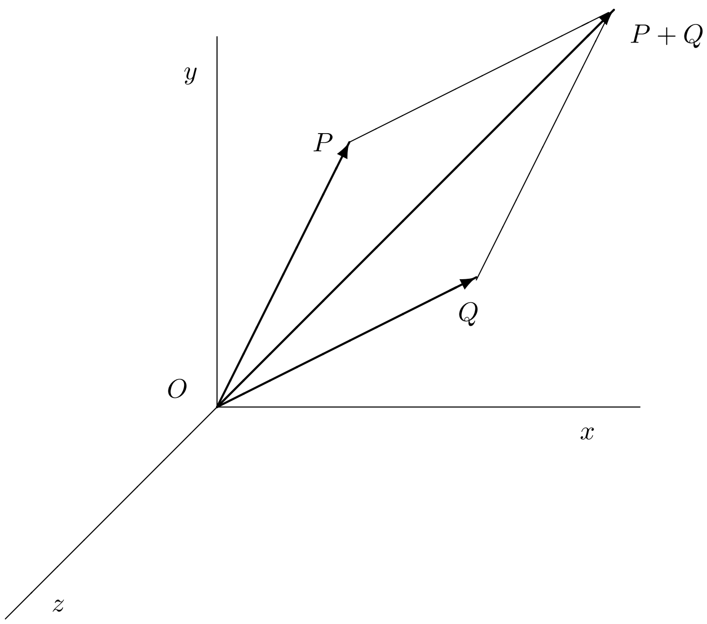
Definition 7 *For any two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$, we define their sum to be*

$$\begin{aligned} \vec{u} + \vec{v} &= (u_1, u_2, u_3) + (v_1, v_2, v_3) \\ &= (u_1 + v_1, u_2 + v_2, u_3 + v_3) \end{aligned}$$

Similarly, we define their difference to be

$$\begin{aligned} \vec{u} - \vec{v} &= (u_1, u_2, u_3) - (v_1, v_2, v_3) \\ &= (u_1 - v_1, u_2 - v_2, u_3 - v_3) \end{aligned}$$

Addition of vectors may be pictured using the 'parallelogram law'.



Example For the following pair of vectors $\vec{u} = (8, 4, -3)$ and $\vec{v} = (-2, 2, 0)$ in $\mathbb{R}^3$, we can calculate, for example

$$\begin{aligned}
\vec{u} + \vec{v} &= (8, 4, -3) + (-2, 2, 0) \\
&= (6, 6, -3) \\
\vec{u} - \vec{v} &= (8, 4, -3) - (-2, 2, 0) \\
&= (10, 2, -3)
\end{aligned}$$

Theorem 4 *(VECTOR ADDITION)*

i *For every* $\vec{u}, \vec{v} \in \mathbb{R}^3$*, we have* $\vec{u} + \vec{v} \in \mathbb{R}^3$*.*

ii *For every* $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$*, we have* $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$*.*

iii *For every* $\vec{u} \in \mathbb{R}^3$*, we have* $\vec{u} + 0 = \vec{u}$ *where* $0 = (0, 0, 0) \in \mathbb{R}^3$*.*

iv *For every* $\vec{u} \in \mathbb{R}^3$*, there exists* $\vec{v} \in \mathbb{R}^3$ *such that* $\vec{u} + \vec{v} = 0$*.*

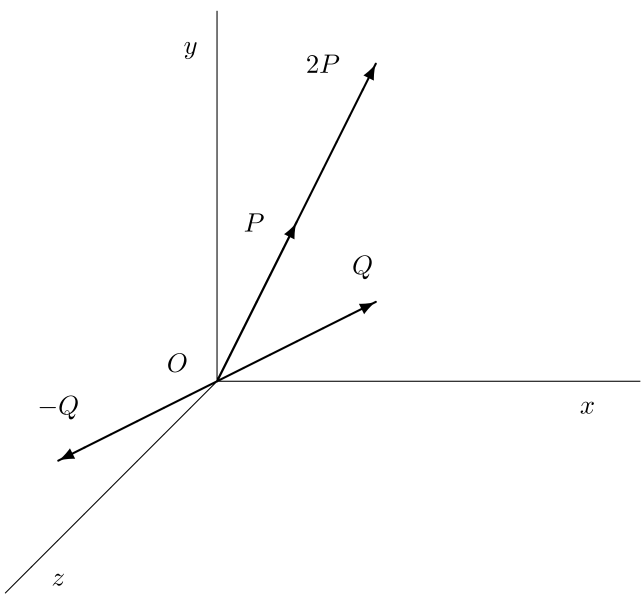
v *For every* $\vec{u}, \vec{v} \in \mathbb{R}^3$*, we have* $\vec{u} + \vec{v} = \vec{v} + \vec{u}$*.*

3.2 Scalar Multiplication of Vectors

Definition 8 *For any vector* $\vec{u} = (u_1, u_2, u_3)$ *in* $\mathbb{R}^3$ *and any scalsr* $c \in \mathbb{R}$*, we define the scalar multiple to be*

$$c\vec{u} = c(u_1, u_2, u_3) = (cu_1, cu_2, cu_3)$$

Scalar multiplication may be pictured as follows –



Example For the following pair of vectors $\vec{u} = (2, 3, -4)$ and $\vec{v} = (-1, 3, 8)$ in $\mathbb{R}^3$, we can evaluate, for example

$$\begin{aligned} 2\vec{u} + 4\vec{v} &= 2(2, 3, -4) + 4(-1, 3, 8) \\ &= (4, 6, -8) + (-4, 12, 32) \\ &= (0, 18, 24) \\ 2\vec{u} - \vec{v} &= 2(2, 3, -4) - (-1, 3, 8) \\ &= (4, 6, -8) - (-1, 3, 8) \\ &= (5, 3, -16) \end{aligned}$$

Theorem 5 *(SCALAR MULTIPLICATION)*

i *For every* $c \in \mathbb{R}$ *and* $\vec{u} \in \mathbb{R}^3$, *we have* $c\vec{u} \in \mathbb{R}^3$.

ii *For every* $c \in \mathbb{R}$ *and* $\vec{u}, \vec{v} \in \mathbb{R}^3$, *we have* $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$.

iii *For every* $a, b \in \mathbb{R}$ *and* $\vec{u} \in \mathbb{R}^3$, *we have* $(a + b)\vec{u} = a\vec{u} + b\vec{u}$.

iv *For every* $a, b \in \mathbb{R}$ *and* $\vec{u} \in \mathbb{R}^3$, *we have* $(ab)\vec{u} = a(b\vec{u})$.

v *For every* $\vec{u} \in \mathbb{R}^3$, *we have* $1\vec{u} = \vec{u}$.

Exercise For the following pair of vectors $\vec{u} = (1, -3, 5)$ and $\vec{v} = (1, -2, 4)$ in $\mathbb{R}^3$, evaluate

i $2\vec{u} + 3\vec{v}$,

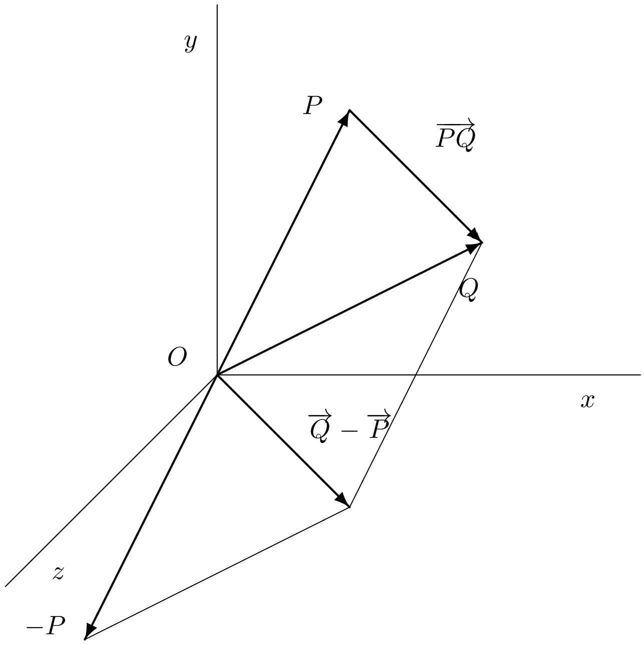
ii $\vec{u} - 5\vec{v}$,

iii $4\vec{u} + 3\vec{v}$.

Remark There is another way in which vectors may be pictured – namely as 'arrows' in three dimensions. The vector (u_1, u_2, u_3) can be pictured by an arrow with initial point O and terminal point (u_1, u_2, u_3) . It is, however, convenient to picture vectors in a more general way. Consider an arrow with the initial point $P = (x_1, y_1, z_1)$ and terminal point $Q = (x_2, y_2, z_2)$. This arrow is denoted by $\overrightarrow{PQ}$. We define

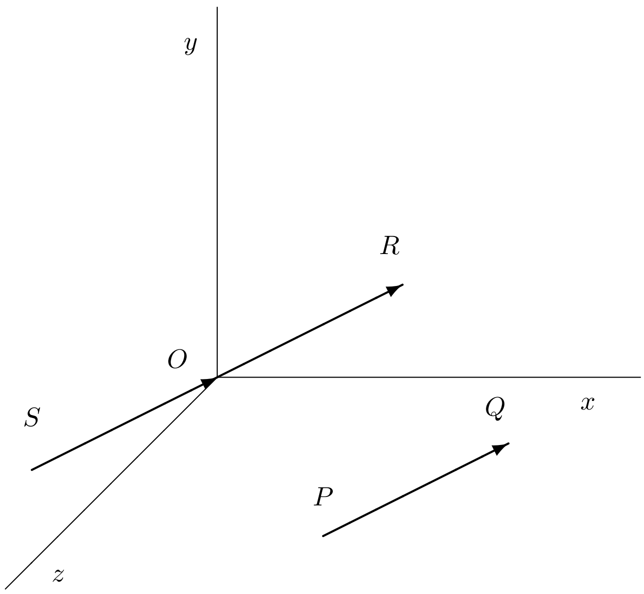
$$\begin{aligned} \overrightarrow{PQ} &= (x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ &= \vec{Q} - \vec{P} \end{aligned}$$

We can picture this as follows – using the parallelogram law



Recall that we may picture a vector as an arrow which has a given length and lies in a given direction, and may be positioned between any pair of points provided that the points determine the same length and direction. The following example illustrates this point.

Example Let $P = (2, -5, 4)$, $Q = (3, -2, 6)$, $R = (1, 3, 2)$ and $S = (-1, -3, -2)$ be four points in space, as shown.



Then

$$
\begin{aligned}
\overrightarrow{PQ} &= (1, 3, 2) \\
\overrightarrow{OR} &= (1, 3, 2) \\
\overrightarrow{SO} &= (1, 3, 2)
\end{aligned}
$$

These arrows represent the same vector, namely, $A = (1, 3, 2)$ and we write

$$\overrightarrow{PQ} = \overrightarrow{OR} = \overrightarrow{SO} = A = (1, 3, 2)$$

Exercise In each case write the vector $\vec{u}$ in terms of components

i $\vec{u}$ is a vector from the point $A(1, -5, 4)$ to the point $B(2, 0, 4)$,

ii $\vec{u}$ is a vector from the point $A(1, 2, 3)$ to the point $B(4, 5, 6)$,

iii $\vec{u}$ is a vector from the point $A(-2, 1, 9)$ to the point $B(3, -6, 8)$.

[**Solution**: $\vec{u} = (1, 5, 0)$, $\vec{u} = (3, 3, 3)$, $\vec{u} = (5, -7, -1)$].

Exercise Let

$$\begin{aligned} \vec{u} &= (2, -1, 3) \\ \vec{v} &= (-4, 2 \cdot 5, 3) \\ \vec{w} &= (1, 1, -2) \end{aligned}$$

Write each of the following in terms of components

i $2\vec{u}$

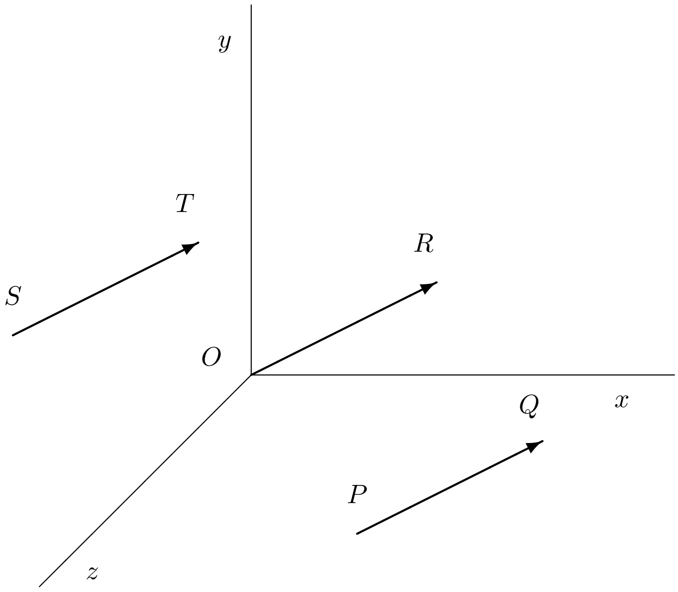
ii $3\vec{u} - 2\vec{v}$

iii $\vec{v} - 2\vec{u} + 4\vec{w}$

iv $2(\vec{u} + \vec{v}) - \vec{w}$

[**Solution**: i $(4, -2, 6)$, ii $(14, -8, 3)$, iii $(-4, 8 \cdot 5, -11)$, iv $(-5, 2, 14)$].

Exercise Let $P = (2, -6, 8)$, $Q = (6, -2, 5)$, $R = (4, 4, -3)$, $S = (-10, 2, 5)$ and $T = (-6, 6, 2)$ be five points in space, as shown.



Show that $\overrightarrow{PQ} = \overrightarrow{OR} = \overrightarrow{ST}$.

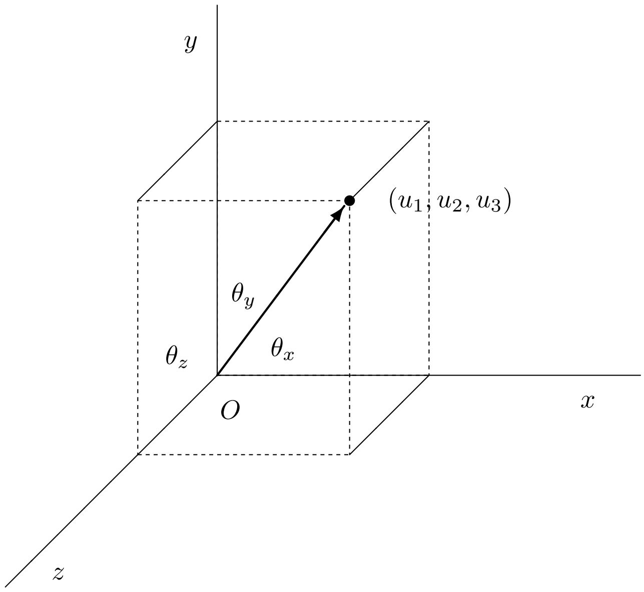
3.3 Magnitude and Direction

Definition 9 *For any vector $\vec{u} = (u_1, u_2, u_3)$ in $\mathbb{R}^3$, we define the magnitude of $\vec{u}$ to be the non-negative real number*

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

The direction of the vector in three-dimensions $\mathbb{R}^3$ in is defined by three angles θ_x, θ_y and θ_z the vector makes with the x-axis, y-axis and z-axis respectively. For a vector $\vec{u} = (u_1, u_2, u_3)$ in $\mathbb{R}^3$

$$\begin{aligned}
\cos\theta_x &= \frac{u_1}{\|\vec{u}\|} \\
\cos\theta_y &= \frac{u_2}{\|\vec{u}\|} \\
\cos\theta_z &= \frac{u_3}{\|\vec{u}\|}
\end{aligned}$$



These angles are difficult to picture, since they are not in the x–plane only or the y–plane only or the z–plane only. For $\vec{u} = (u_1, u_2, u_3)$ in $\mathbb{R}^3$, we can write,

$$\begin{aligned} u_1 &= \|\vec{u}\| \cos\theta_x \\ u_2 &= \|\vec{u}\| \cos\theta_y \\ u_3 &= \|\vec{u}\| \cos\theta_z \end{aligned}$$

These equations will determine the components u_1, u_2 and u_3 of the vector $\vec{u} = (u_1, u_2, u_3)$ in $\mathbb{R}^3$ when presented with the magnitude and direction of $\vec{u}$ only.

Furthermore

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

Remark A vector of magnitude 1 is called a *unit vector* or *normalised vector*. Any non-zero vector determines a unit vector

$$\begin{aligned} \frac{1}{\|\vec{u}\|}\vec{u} &= \left(\frac{u_1}{\|\vec{u}\|}, \frac{u_2}{\|\vec{u}\|}, \frac{u_3}{\|\vec{u}\|}\right) \\ \frac{1}{\|\vec{u}\|}\vec{u} &= (\cos\theta_x, \cos\theta_y, \cos\theta_z) \end{aligned}$$

This normalised form may be used to conveniently calculate the angles $\theta_x, \theta_y, \theta_z$ for a given vector $\vec{u} = (u_1, u_2, u_3)$ in $\mathbb{R}^3$. This form is usually used to describe the direction of any vector in $\mathbb{R}^3$.

Example We can write $\vec{v} = (2,3,4)$ as a unit vector (or normalised vector) along $\vec{u}$

$$\frac{1}{\|\vec{u}\|}\vec{u} = \left(\frac{u_1}{\|\vec{u}\|}, \frac{u_2}{\|\vec{u}\|}, \frac{u_3}{\|\vec{u}\|}\right)$$

Now $\|\vec{u}\| = \sqrt{29}$. Hence

$$\frac{1}{\|\vec{u}\|}\vec{u} = \left(\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right) = (\cos\theta_x, \cos\theta_x, \cos\theta_x)$$

It follows that

$$\begin{aligned}
\cos\theta_x &= \left(\frac{2}{\sqrt{29}}\right) = 0\cdot 3714\\
\cos\theta_y &= \left(\frac{3}{\sqrt{29}}\right) = 0\cdot 5571\\
\cos\theta_z &= \left(\frac{4}{\sqrt{29}}\right) = 0\cdot 7428
\end{aligned}$$

Hence

$$\begin{aligned}
\theta_x &= 68\cdot 20\\
\theta_y &= 56\cdot 15\\
\theta_z &= 42\cdot 03
\end{aligned}$$

Exercise For each of the following vectors in $\mathbb{R}^3$, find the angles $\theta_x, \theta_y, \theta_z$, i.e., the angle the vector makes with the x–axis, y–axis and z–axis respectively.

i $\vec{u} = (-2,1,1)$

ii $\vec{u} = (1,-1,1)$

iii $\vec{u} = (-4,-2,2)$

[**Solution**: i $\theta_x = 144\cdot 74^o$, $\theta_y = 65\cdot 91^o$, $\theta_z = 65\cdot 91^o$
ii $\theta_x = 54\cdot 74^o$, $\theta_y = 125\cdot 26^o$, $\theta_z = 54\cdot 74^o$
iii $\theta_x = 144\cdot 74^o$, $\theta_y = 114\cdot 09^o$, $\theta_z = 65\cdot 91^o$].

Remark A very common problem in games is that an object moves a distance in a particular direction and we have to determine where it ends up, so that we can draw it again in the new position. Suppose that an object is at position $A(1,2,3)$ in one frame and it moves 10 units in the direction $\theta_x = 75^o$, $\theta_y = 50^o$ and $\theta_z = 43\cdot 86^o$ before the next frame. What is the new position in the second frame?

Let $\vec{u} = (u_1, u_2, u_3)$ with,

$$\begin{aligned} u_1 &= \|\vec{u}\| \cos\theta_x = 10\cos 75^o \\ u_2 &= \|\vec{u}\| \cos\theta_y = 10\cos 50^o \\ u_3 &= \|\vec{u}\| \cos\theta_z = 10\cos 43\cdot 86^o \end{aligned}$$

Hence $\vec{u} = (2\cdot 588, 6\cdot 43, 7\cdot 21)$. The new position in the second frame will be

$$\begin{aligned} A + \vec{u} &= (1,2,3) + (2\cdot 588, 6\cdot 43, 7\cdot 21) \\ &= (3\cdot 588, 8\cdot 43, 10\cdot 21) \end{aligned}$$

Exercise For each of the following objects defined in frame one by a vector A, find its new position in frame two:

i The object starts from $A(2,3,1)$ and moves 7 units in the direction $\theta_x = 90^o$, $\theta_y = 130^o$ and $\theta_z = 40^o$.

ii The object starts from $A(1,2,3)$ and moves $2\cdot 8$ units in the direction $\theta_x = 120^o$, $\theta_y = 60^o$.

iii The object starts from $A(-1,1,2)$ and moves $1\cdot 5$ units in the direction $\theta_y = 50^o$, $\theta_y = 70^o$.

[**Solution**: i $(2, -1\cdot 50, 6\cdot 36)$
ii $(-0\cdot 4, 3\cdot 4, 4\cdot 98)$
iii $(0\cdot 03, 1\cdot 96, 2\cdot 51)$].

3.4 The Scalar Product

Definition 10 *Suppose that* $\vec{u} = (u_1, u_2, u_3)$ *and* $\vec{v} = (v_1, v_2, v_3)$ *are vectors in* $\mathbb{R}^3$ *and that* $\theta \in [0, \pi]$ *represents the angle between them. We define the scalar product* $\vec{u}.\vec{v}$ *of* $\vec{u}$ *and* $\vec{v}$ *by*

$$\vec{u}.\vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$$

Alternatively, we can write

$$\vec{u}.\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

Theorem 6 *(SCALAR PRODUCT)*

Suppose that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ *and* $c \in \mathbb{R}$*, then*

i $\vec{u}.\vec{v} = \vec{v}.\vec{u}$

ii $\vec{u}.(\vec{v} + \vec{w}) = (\vec{u}.\vec{v}) + (\vec{u}.\vec{w})$

iii $c(\vec{u}.\vec{v}) = (c\vec{u}).\vec{v} = \vec{u}.(c\vec{v})$

iv $\vec{u}.\vec{u} \geq 0$

v $\vec{u}.\vec{u} = 0$ *if and only if* $\vec{u} = 0$

Remark

i The *scalar product* is also known as the *dot product* or the *inner product* of $\vec{u}$ and $\vec{v}$.

ii We say that two non-zero vectors in $\mathbb{R}^3$ are *orthogonal* if the angle between them is $\frac{\pi}{2}$. It follows immediately from the definition of scalar product that any two non-zero vectors $\vec{u}, \vec{v} \in \mathbb{R}^3$ are orthogonal if and only if $\vec{u}.\vec{v} = 0$.

iii Using the definition of scalar product we can calculate the angle between $\vec{u}$ and $\vec{v}$ since

$$\cos\theta = \frac{\vec{u}.\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

Example Suppose $\vec{u} = (2, 4, 6)$ and $\vec{v} = (1, -2, 3)$. Then

$$\begin{aligned} \vec{u}.\vec{v} &= u_1v_1 + u_2v_2 + u_3v_3 \\ &= 2.1 + 4.(-2) + 6.3 \\ &= 12 \end{aligned}$$

Example Suppose $\vec{u} = (2,0,0)$ and $\vec{v} = (1,1,\sqrt{2})$. Then we have $\vec{u}.\vec{v} = 2$. Note now that $\|\vec{u}\| = 2$ and $\|\vec{v}\| = 2$. It follows that

$$\cos\theta = \frac{\vec{u}.\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{2}{4} = \frac{1}{2}$$

Hence

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^o$$

Example Suppose $\vec{u} = (-4,-1,1)$ and $\vec{v} = (1,-2,5)$. Then we have $\vec{u}.\vec{v} = 3$. Note now that $\|\vec{u}\| = \sqrt{18}$ and $\|\vec{v}\| = \sqrt{30}$. It follows that

$$\cos\theta = \frac{\vec{u}.\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{3}{\sqrt{18}.\sqrt{30}} = \frac{3}{23\cdot 24}$$

Hence

$$\theta = \cos^{-1}\left(\frac{3}{23\cdot 24}\right) = 82\cdot 58^o$$

Example Suppose $\vec{u} = (2,3,5)$ and $\vec{v} = (1,1,-1)$. Then we have $\vec{u}.\vec{v} = 0$. It follows that $\vec{u}$ and $\vec{v}$ are orthogonal.

Exercise Let $\vec{u} = (2,4,-3)$ and $\vec{v} = (8,-1,-1)$. Determine the *scalar product* $\vec{u}.\vec{v}$ and hence determine the angle between $\vec{u}$ and $\vec{v}$.

Exercise Let $\vec{u} = (1,-2,-5)$ and $\vec{v} = (0,-1,-1)$. Determine the *scalar product* $\vec{u}.\vec{v}$ and hence determine the angle between $\vec{u}$ and $\vec{v}$.

Exercise Let $\vec{u} = (5,6,-9)$ and $\vec{v} = (1,-1,-1)$. Determine the *scalar product* $\vec{u}.\vec{v}$ and hence determine the angle between $\vec{u}$ and $\vec{v}$.

3.5 Components and Projections

Definition 11 *Let $\vec{u}$ and $\vec{v}$ be two non-zero vectors and θ the angle between them. The scalar component of $\vec{u}$ along $\vec{v}$ is the number*

$$\frac{\vec{u}.\vec{v}}{\|v\|}$$

** Since $\vec{u}.\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$, the scalar component may also be written as $\|\vec{u}\|\cos\theta$.*

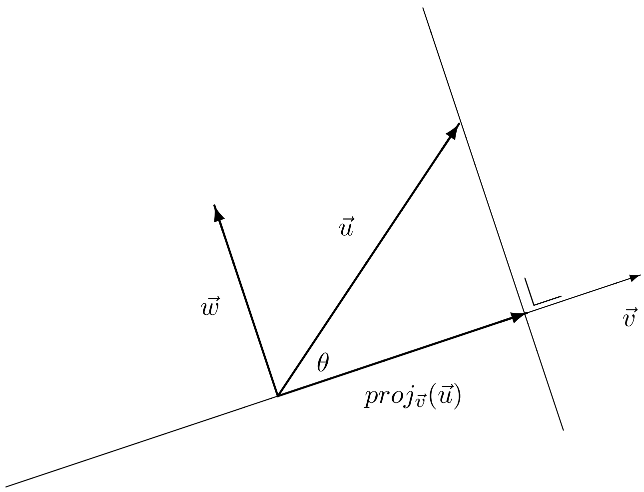
The vector projection of $\vec{u}$ along $\vec{v}$, denoted by $proj_{\vec{v}}(\vec{u})$, is the vector defined by

$$proj_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u}.\vec{v}}{\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|}$$

that is, the scalar multiple of the direction of $\vec{v}$ by the scalar component of $\vec{u}$ along $\vec{v}$.

Remark

i In some applications it can be useful to decompose or resolve a vector $\vec{u}$ into two vectors – one parallel to non-zero vector $\vec{v}$ and the other perpendicular to $\vec{v}$.



To resolve a given vector $\vec{u}$ into two vectors – one parallel to to a given non-zero vector $\vec{v}$ and the other perpendicular to the vector $\vec{v}$ we first calculate $proj_{\vec{v}}(\vec{u})$, the *vector projection* of $\vec{u}$ along $\vec{v}$ and secondly a perpendicular vector to $\vec{v}$, which we will label $\vec{w}$ and is given as

$$\vec{w} = \vec{u} - proj_{\vec{v}}(\vec{u})$$

The resolvent of $proj_{\vec{v}}(\vec{u})$ and $\vec{w}$ will yield $\vec{u}$. Furthermore, their *scalar product* is zero.

ii As the name suggests, the *scalar component* is a scalar and the *vector projection* is a vector.

iii The $proj_{\vec{v}}(\vec{u})$ has the same direction as $\vec{v}$ if θ is acute, and the opposite direction if θ is obtuse.

iv The length or magnitude of $proj_{\vec{v}}(\vec{u})$ is

$$\left|\frac{\vec{u}.\vec{v}}{\|\vec{v}\|}\right|$$

that is, the absolute value of the scalar component of $\vec{u}$ along $\vec{v}$.

Example Let $\vec{u} = (2, -1, 3)$ and $\vec{v} = (1, -3, -1)$.

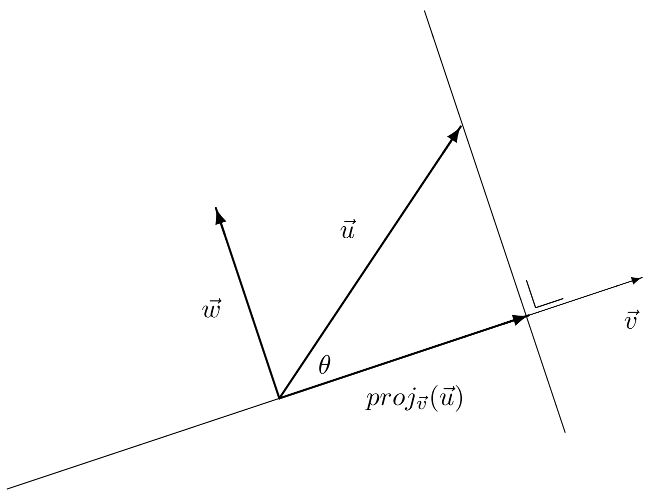
To determine the *vector projection* of $\vec{u}$ along $\vec{v}$ we have

$$proj_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u}.\vec{v}}{\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|}$$

Hence

$$proj_{\vec{v}}(\vec{u}) = \left(\frac{2+3-3}{\sqrt{11}}\right).\frac{(1,-3,-1)}{\sqrt{11}} = \frac{2}{11}(1,-3,-1)$$

Note Consider the example above with the diagram depicting a vector projection. From the vector $\vec{u}$ we have determined $proj_{\vec{v}}(\vec{u})$, the vector projection of $\vec{u}$ along $\vec{v}$.



We can determine the vector $\vec{w}$ as follows

$$\vec{w} = \vec{u} - proj_{\vec{v}}(\vec{u}) = (2,-1,3) - \frac{2}{11}(1,-3,-1) = \frac{5}{11}(4,-1,7)$$

The resolvent of $proj_{\vec{v}}(\vec{u})$ and $\vec{w}$ should yield $\vec{u}$.

$$proj_{\vec{v}}(\vec{u}) + \vec{w} = \frac{2}{11}(1,-3,-1) + \frac{5}{11}(4,-1,7) = \left(\frac{22}{11}, \frac{-11}{11}, \frac{33}{11}\right) = (2,-1,3)$$

Finally, their *scalar product* is zero.

$$proj_{\vec{v}}(\vec{u}).\vec{w} = \frac{2}{11}(1,-3,-1).\frac{5}{11}(4,-1,7) = \frac{40}{11} + \frac{30}{11} - \frac{70}{11} = 0$$

as expected.

Exercise Let $\vec{u} = (2, 0, 1)$ and $\vec{v} = (3, 1, -2)$. Determine the *vector projection* of $\vec{u}$ along $\vec{v}$

Exercise Let $\vec{u} = (-1, 2, 4)$ and $\vec{v} = (0, 1, -6)$. Determine the *vector projection* of $\vec{u}$ along $\vec{v}$

Exercise Let $\vec{u} = (2, 2, 7)$ and $\vec{v} = (3, 6, -5)$. Determine the *vector projection* of $\vec{u}$ along $\vec{v}$

Exercise Let $\vec{u} = (2, 1, 2)$ and $\vec{v} = (6, -1, 0)$. Resolve the vector $\vec{u}$ into vectors parallel and perpendicular to the vector $\vec{v}$.

Exercise Let $\vec{u} = (3, 4, 5)$ and $\vec{v} = (1, 1, -2)$. Resolve the vector $\vec{u}$ into vectors parallel and perpendicular to the vector $\vec{v}$.

3.6 The Vector Product

We now discuss a product of vectors unique to $\mathbb{R}^3$. The idea of vector products has a wide applications in geometry, physics and engineering, and is motivated by the wish to find a vector that is perpendicular to two given vectors.

Definition 12 *Suppose that $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$ and that $\theta \in [0, \pi]$ represents the angle between them. Let $\vec{n}$ be a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$. Then the vector product (or cross product) of $\vec{u}$ and $\vec{v}$ is the vector denoted by $\vec{u} \times \vec{v}$ and defined by*

$$\vec{u} \times \vec{v} = ||\vec{u}||||\vec{v}|| \sin\theta \vec{n}$$

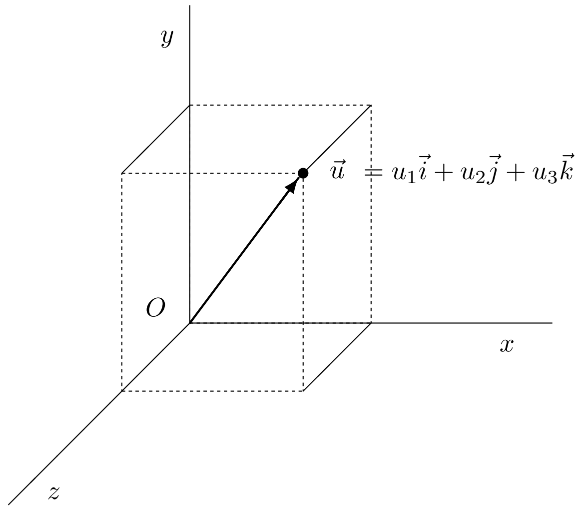
Alternatively

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

Remark The vector product $\vec{u} \times \vec{v}$ yields a vector in $\mathbb{R}^3$. In order to develop this component representation of $\vec{u} \times \vec{v}$ we will switch momentarily from the component representation of a vector $\vec{u} = (u_1, u_2, u_3)$ to its equivalent cartesian form

$$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$$

where the three unit vectors $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$ and $\vec{k} = (0, 0, 1)$, the unit vectors along the x,y and z-axes respectively.



Suppose

$$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$$
$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

$$\begin{aligned}
\vec{u} \times \vec{v} &= (u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) \times (v_1\vec{i} + v_2\vec{j} + v_3\vec{k}) \\
&= u_1v_1\vec{i} \times \vec{i} + u_1v_2\vec{i} \times \vec{j} + u_1v_3\vec{i} \times \vec{k} \\
&\quad + u_2v_1\vec{j} \times \vec{i} + u_2v_2\vec{j} \times \vec{j} + u_2v_3\vec{j} \times \vec{k} \\
&\quad\quad + u_3v_1\vec{k} \times \vec{i} + u_3v_2\vec{k} \times \vec{j} + u_3v_3\vec{k} \times \vec{k}
\end{aligned}$$

Using each of the following facts

$$\vec{u} \times \vec{v} = ||\vec{u}||||\vec{v}|| \sin\theta\vec{n}$$
$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$

we can establish each of the following

$$\begin{array}{lll}
\vec{i} \times \vec{i} = 0 & \vec{i} \times \vec{j} = \vec{k} & \vec{j} \times \vec{i} = -\vec{k} \\
\vec{j} \times \vec{j} = 0 & \vec{j} \times \vec{k} = \vec{i} & \vec{i} \times \vec{k} = -\vec{j} \\
\vec{k} \times \vec{k} = 0 & \vec{k} \times \vec{i} = \vec{j} & \vec{k} \times \vec{j} = -\vec{i}
\end{array}$$

Hence, we have

$$\begin{aligned}
\vec{u} \times \vec{v} &= u_1v_1(0) + u_1v_2\vec{k} + u_1v_3(-\vec{j}) \\
&\quad + u_2v_1(-\vec{k}) + u_2v_2(0) + u_2v_3\vec{i} \\
&\quad\quad + u_3v_1\vec{j} + u_3v_2(-\vec{i}) + u_3v_3(0) \\
&= (u_2v_3 - u_3v_2)\vec{i} + (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}
\end{aligned}$$

Finally, returning to our component representation we have

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

Remark A convenient way of determining the vector product $\vec{u} \times \vec{v}$ is as follows

$$\vec{u} \times \vec{v} = det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

Using the cofactor expansion by row 1, we have

$$\vec{u} \times \vec{v} = det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} \vec{i} - det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} \vec{j} + det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \vec{k}$$

$$= \left(det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix}, -det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix}, det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \right)$$

$$= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

We will first show that the vector product $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

Theorem 7

Suppose that $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$. Then

i $\vec{u}.(\vec{u} \times \vec{v}) = 0$

ii $\vec{v}.(\vec{u} \times \vec{v}) = 0$

Example Suppose that $\vec{u} = (1, -1, 2)$ and $\vec{v} = (3, 0, 2)$. Then

$$\vec{u} \times \vec{v} = det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 3 & 0 & 2 \end{pmatrix}$$

$$= \left(det \begin{pmatrix} -1 & 2 \\ 0 & 2 \end{pmatrix}, -det \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, det \begin{pmatrix} 1 & -1 \\ 3 & 0 \end{pmatrix} \right)$$

$$= (-2 + 0, -(2 - 6), 0 + 3)$$

$$= (-2, 4, 3)$$

Note that $(1, -1, 2).(-2, 4, 3) = 0$ and $(3, 0, 2).(-2, 4, 3) = 0$.

Exercise For the vectors $\vec{u} = (1, 2, 3)$ and $\vec{v} = (3, 2, 1)$ in $\mathbb{R}^3$, evaluate

i $\vec{u} \times \vec{v}$

ii $\vec{v} \times \vec{u}$

What comment can you make about you answer.

Theorem 8 *(VECTOR PRODUCT)*

Suppose that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$. Then

i $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$;

ii $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$;

iii $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$;

iv $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$;

v $\vec{u} \times 0 = 0$;

vi $\vec{u} \times \vec{u} = 0$.

Now to consider an application of vector product – to evaluate the area of a parallelogram. To do this we first establish the following result.

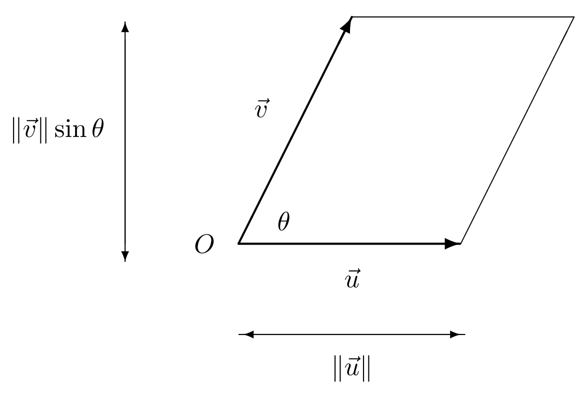
Theorem 9

Suppose that $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are non-zero vectors in $\mathbb{R}^3$, and that $\theta \in [0, \pi]$ represents the angle between them. Then

i $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u}.\vec{v})^2$

ii $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$

Now consider a parallelogram below.



The base of the parallelogram is given by $\|\vec{u}\|$, and hence the height of the parallelogram is given as $\|\vec{v}\|\sin\theta$. Therefore, from theorem 6 we can say that the area of the parallelogram is given by $\|\vec{u}\times\vec{v}\|$.

Theorem 10

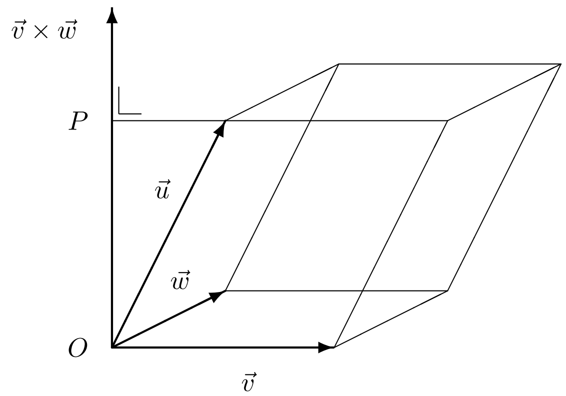
Suppose that $\vec{u},\vec{v}\in\mathbb{R}^3$. Then the parallelogram with $\vec{u}$ and $\vec{v}$ as two of its sides has area $\|\vec{u}\times\vec{v}\|$.

Exercise Let $\vec{u}=(1,1,-4)$ and $\vec{v}=(4,1,7)$ in $\mathbb{R}^3$.

Determine the area of the *parallelogram* that is defined by $\vec{u}$ and $\vec{v}$.

3.7 Scalar Triple Product

Now suppose that $\vec{u},\vec{v},\vec{w}\in\mathbb{R}^3$ that do **not** lie all on the same plane. What is formed is a parallelepiped, i.e. a solid body in which each face is a parallelogram, with $\vec{u},\vec{v}$ and $\vec{w}$ as three of its edges.



The base of the parallelepiped has area $\|\vec{v}\times\vec{w}\|$

If the vector OP is perpendicular to the base of the parallelepiped, then OP is in the direction of $\vec{v}\times\vec{w}$. Now the height of the parallelepiped is equal to the norm of the orthogonal projection of $\vec{u}$ on $\vec{v}\times\vec{w}$. In other words, the parallelepiped has height

$$proj_{\vec{v}\times\vec{w}}(\vec{u})=\left(\frac{\vec{u}.(\vec{v}\times\vec{w})}{\|\vec{v}\times\vec{w}\|}\right)\frac{\vec{v}\times\vec{w}}{\|\vec{v}\times\vec{w}\|}$$

Hence, we have

$$\|proj_{\vec{v}\times\vec{w}}(\vec{u})\|=\frac{\vec{u}.(\vec{v}\times\vec{w})}{\|\vec{v}\times\vec{w}\|}$$

Therefore the volume of the parallelepiped is given by

$$V = \vec{u}.(\vec{v} \times \vec{w})$$

Theorem 11

Suppose that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$. Then the parallelepiped with $\vec{u}$, $\vec{v}$ and $\vec{w}$ as three of its edges has volume $\vec{u}.(\vec{v} \times \vec{w})$.

Definition 13 *Suppose that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$. Then $\vec{u}.(\vec{v} \times \vec{w})$ is called the scalar triple product of $\vec{u}$, $\vec{v}$ and $\vec{w}$.*

Remark It follows from theorem 9 that three vectors in $\mathbb{R}^3$ are *coplanar* if and only if their scalar triple product is zero.

Example Suppose that $\vec{u} = (1, 0, 1)$, $\vec{v} = (2, 1, 3)$ and $\vec{w} = (0, 1, 1)$. Then

$$\vec{u}.(\vec{v} \times \vec{w}) = det \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix} = 0$$

Hence $\vec{u}$, $\vec{v}$ and $\vec{w}$ are coplanar.

Example The volume of the parallelepiped with $\vec{u} = (1, 0, 1)$, $\vec{v} = (2, 1, 4)$ and $\vec{w} = (0, 1, 1)$ as three of its edges are given by

$$V = \vec{u}.(\vec{v} \times \vec{w}) = det \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix} = -1$$

We take the absolute value 1.

Exercise Let $\vec{u} = (2, 1, 3)$, $\vec{v} = (4, -1, 0)$ and $\vec{w} = (2, 0, 1)$.

Show that $\vec{u}, \vec{v}$ and $\vec{w}$ are *coplanar*.

Exercise Let $\vec{u} = (5\lambda, 2\lambda, 3)$, $\vec{v} = (1, 1, 0)$ and $\vec{w} = (0, 2, -1)$.

For which values of λ are these vectors *coplanar*?

3.8 Some Exercises

Exercise Consider the following vectors $\vec{u} = (1, 2, 3)$ and $\vec{v} = (3, 2, 1)$ in $\mathbb{R}^3$.

i Evaluate $\vec{u} - 4\vec{v}$

ii Evaluate $\vec{u}.\vec{v}$

ii Determine $||7\vec{u} - 2\vec{v}||$

iv Determine $\vec{u} \times \vec{v}$.

v Determine the *vector projection* of $\vec{u}$ along $\vec{v}$, i.e., $proj_{\vec{v}}(\vec{u})$

Exercise Consider the following vectors $\vec{u} = (1, 0, 1)$, $\vec{v} = (2, 1, 3)$ and $\vec{w} = (0, 1, 1)$ in $\mathbb{R}^3$.

i Evaluate $\vec{u} - 4\vec{v} + 2\vec{w}$

ii Determine $||7\vec{u} - 2\vec{v}||$

iii Evaluate $\vec{u}.\vec{v}$

iv Determine the angle θ between $\vec{u}$ and $\vec{v}$.

v Determine the *vector projection* of $\vec{u}$ along $\vec{v}$, i.e., $proj_{\vec{v}}(\vec{u})$

vi Calculate the components a, b and c of some non-zero vector that is orthogonal to $\vec{u}$ and $\vec{v}$.

vii Determine the area of the *parallelogram* that is defined by $\vec{u}$ and $\vec{v}$.

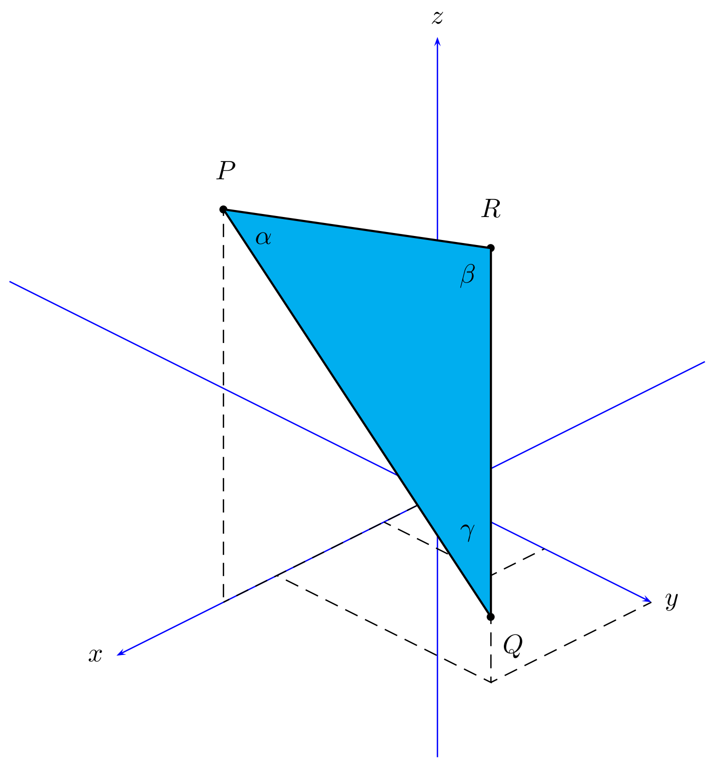
viii Determine $\vec{u}.(\vec{v} \times \vec{w})$. What comment can you make about the vectors $\vec{u}, \vec{v}$ and $\vec{w}$?

Exercise Find the interior angles α, β, γ of a triangle ABC whose vertices are the points

$$A(-1, 0, 2) \quad , \quad B(2, 1, -1) \quad , \quad C(1, -2, 2)$$

Exercise Consider the following points in $\mathbb{R}^3$.

$$P(1,2,-3) \quad , \quad Q(3,2,1) \quad , \quad R(4,0,-2)$$



i Evaluate $P - 2Q + 3R$.

ii Determine $||2P - 4Q||$.

iii Evaluate $P.Q$, i.e., the *scalar product* of P with Q.

iv Evaluate $P \times Q$.

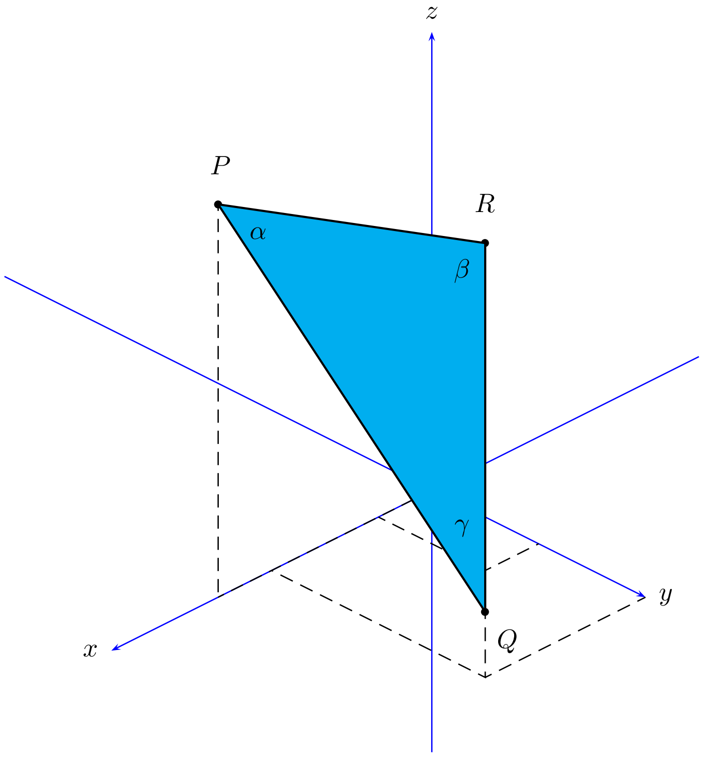
v Show that the vectors defining this triangle $\triangle PQR$ are *coplanar*.

vi Determine all internal angles α, β, γ of the triangle $\triangle PQR$.

vii Determine the area of the triangle $\triangle PQR$.

Exercise Consider the following points in $\mathbb{R}^3$.

$$P(2,3,4) \quad , \quad Q(1,2,-1) \quad , \quad R(4,10,-4)$$



i Evaluate $P - 2Q + 3R$.

ii Determine $||2P - 4Q||$.

iii Evaluate $P.Q$, i.e., the *scalar product* of P with Q.

iv Evaluate $P \times Q$.

v Show that the vectors defining this triangle $\triangle PQR$ are *coplanar*.

vi Determine all internal angles α, β, γ of the triangle $\triangle PQR$.

vii Determine the area of the triangle $\triangle PQR$.