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BASIC TRIGONOMETRY

1 Basic Trigonometry

These definitions are required in the design of simple graphics programs and in all computer games.

1.1 Angular Measure

The most common system is *degree measure* in which the complete circle is divided into 360 degrees. For more accurate measurement each degree is divided into 60 minutes and each minute is divided into 60 seconds. So, for example

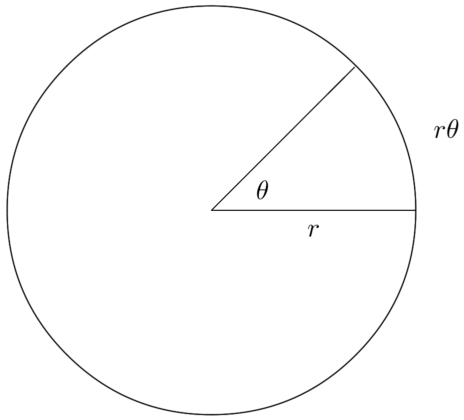
$$25^o30^{'} = 25 \cdot 5^o$$

$$38^o15^{'} = 38 \cdot 25^o$$

$$90^o15^{'}25^{''} = 90 \cdot 2569444^o$$

A mathematically more natural unit of degree measure is the *radian*.

Definition One radian is the angle at the centre of a circle subtended by an arc whose length is equal to the radius.



$$Arc\ Length = r\theta$$

From the diagram and the above definition

$$\frac{r\theta}{r} = \theta = 1\ radian$$

The length of the circumference of a circle is given as $2\pi r$. Hence

$$\frac{2\pi r}{r} = 2\pi\ radians$$

defines the number of radians in the complete circle. This gives us a relationship between the angle measure of degrees and radians.

$$\begin{aligned} 360^o &= 2\pi\ radians \\ 180^o &= \pi\ radians \quad ** \end{aligned}$$

Exercise Convert each of the following angles in degrees to radians:

$$90^o \qquad 30^o \qquad 270^o \qquad 15 \cdot 382^o$$

Exercise Convert each of the following angles in radians to degrees:

$$\frac{\pi}{2} \qquad \frac{\pi}{4} \qquad \frac{5\pi}{6} \qquad 1 \cdot 4\pi \qquad 1\ radian \qquad 2 \cdot 9\ radians$$

Note All computer languages use only *radian measure*. You as a programmer will have to convert from degrees to radians before any calculations in your programme.

Example The following code is written in *C#* **2010 Express Edition** (using a button on a form) and it takes an angle in degrees from textbox 1 and will write the angle in radians out to textbox2 when the button is pressed:

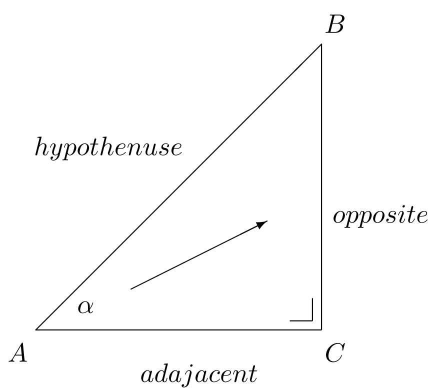
```
private void button1_Click(object sender, EventArgs e)
{
double aAngleInDegrees;
double aAngleInRadians;
aAngleInDegrees = Convert.ToDouble(textBox1.Text);
aAngleInRadians = aAngleInDegrees * Math.PI/180;
textBox2.Text = Convert.ToString(aAngleInRadians);
textBox2.Visible = true;
}
```

Note that the command **aAngleInDegrees * Math.PI / 180** is this segment of code simply converts from degrees to radians since

$$1^o = \frac{\pi}{180}\ radians$$

1.2 Basic Trigonometrical Functions

We can define the three trigonometric functions $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ by use of a right-angled triangle. Consider the triangle ABC



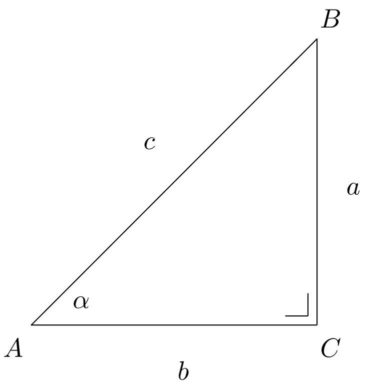
$$\sin\alpha = \frac{opposite}{hypothenuse}$$

$$\cos\alpha = \frac{adjacent}{hypothenuse}$$

$$\tan\alpha = \frac{opposite}{adjacent} = \frac{\sin\alpha}{\cos\alpha}$$

Theorem 1 *(Pythagoras' theorem) In a right-angled triangle , the sum of the squares of the lengths of the sides containing the right angle is equal to the square of the hypothenuse; i.e.*

$$a^2 + b^2 = c^2$$



Three positive integers a, b and c such that $a^2 + b^2 = c^2$ are called *Pythagorean triples.* For example $(3, 4, 5)$, $(5, 12, 13)$, $(6, 8, 10)$, $(8, 15, 17)$, $(9, 12, 15)$ are all solutions of the equation

$$a^2 + b^2 = c^2$$

Remark In the early 1600's, Pierre de Fermat (1601–1665), a French lawyer and mathematician posed the following question – if the power of 2 in the above equation was replaced by 3 could there be found three non-zero integers a, b and c that satisfy the equation $a^3 + b^3 = c^3$? The same question could be asked if the power was increased to 4 then to 5 and down to any positive integer n.

$$
\begin{aligned}
a^3 + b^3 &= c^3 \\
a^4 + b^4 &= c^4 \\
&\dots \\
&\dots \\
a^n + b^n &\neq c^n
\end{aligned}
$$

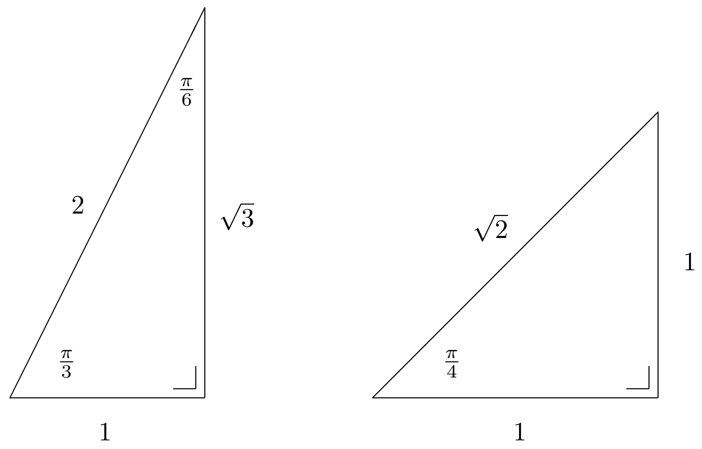
Fermat stated that the no matter how hard you try you will never find integer solutions to these equations. This famous statement become known as Fermat's 'Last' Theorem, which was not solved until 1994 by British-American mathematician Andrew Wiles. Wiles devoted seven years of his life to proving the famous theorem, which may have generated more attempts at proofs than any other theorem.

Pierre de Fermat (1601–1665)

Fermat's 'Last' Theorem states that $a^n + b^n = c^n$ has no non-zero integer solutions for a, b and c when $n > 2$. Fermat stated his theorem in 1637 when he wrote that "I have a truly marvelous" proof of this proposition which this margin is too narrow to contain". Today, however, we believe that Fermat had no such proof.

Remark Recall that there are simple exact expressions for the sine and cosine of the angles

$$\frac{\pi}{6} \ , \ \frac{\pi}{4} \ , \ \frac{\pi}{3}$$



It follows from these right-angled triangles that

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \quad , \quad \sin\frac{\pi}{6} = \frac{1}{2} \quad , \quad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$\cos\frac{\pi}{3} = \frac{1}{2} \quad , \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad , \quad \tan\frac{\pi}{3} = \sqrt{3}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad , \quad \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad , \quad \tan\frac{\pi}{4} = 1$$

Note

i The *sine*, *cosine* and *tangent* of an angle may be calculated on a scientific calculator. If the angle measure is in *degrees* your calculator must be in degree mode (**deg**). Most calculators will default to this mode in order to proceed. If the angle measure is in *radians* your calculator must be in radian mode (**rad**) in order to proceed.

ii All scientific calculators *inverse sine*, *inverse cosine* and *inverse tangent* functions. These trigonometrical functions are denoted as

$$\sin^{-1} \quad , \quad \cos^{-1} \quad , \quad \tan^{-1}$$

Exercise Calculate each of the following, writing each answer accurate to four places of decimals:

$$\sin 42 \cdot 38^o \qquad \sin \frac{2\pi}{3} \qquad \sin 50^o \qquad \sin(1 \cdot 6)\ radians$$

[**Solution**: *0·6740* , *0·8660* , *0·7660* , *0·9996*].

Exercise Find the value of A, accurate to 3 decimal places in each of the following equations:

$$2\cos 3B = A \quad , \quad when\ B = 10^o$$

$$2A\tan 5B = 0 \cdot 5 \quad , \quad when\ B = 1\ radian$$

$$2\sin(3B + \frac{\pi}{3}) = 5A \quad , \quad when\ B = \frac{\pi}{6}$$

[**Solution**: *1·732* , *-0·074* , *0·200*].

To illustrate the use of an *inverse trigonometrical function* consider the following example:

Example To determine the angle α when

$$2\sin 3\alpha = 0 \cdot 5\cos\frac{\pi}{4}$$

we have

$$\begin{aligned} 2\sin 3\alpha &= 0 \cdot 5\cos\frac{\pi}{4} \\ 2\sin 3\alpha &= 0 \cdot 35355 \\ \sin 3\alpha &= 0 \cdot 17678 \\ 3\alpha &= \sin^{-1} 0 \cdot 17678 \\ 3\alpha &= 0 \cdot 17777\ rad \quad (10 \cdot 182^o) \\ \therefore \quad \alpha &= 0 \cdot 05924\ rad \quad (3 \cdot 394^o) \end{aligned}$$

Exercise Find the angle α, giving your answer in degrees accurate to 2 decimal places, in each of the following equations:

$$2\sin 3\alpha = 1 \cdot 4\tan 40^o$$

$$\tan\left(2\alpha - \frac{\pi}{2}\right) = 5 \cdot 2$$

[**Solution**: *11·99°* , *84·56°*].

1.3 Graphics in $C\#$

On the computer screen the origin $(0, 0)$ is at the **top left hand corner** of the screen.

Note

i The trigonometrical functions available in $C\#$ are:

$$\begin{aligned} \textbf{Math.Sin}() &\quad .... \quad \textit{the angle must be in radians} \\ \textbf{Math.Cos}() &\quad .... \quad \textit{the angle must be in radians} \\ \textbf{Math.Tan}() &\quad .... \quad \textit{the angle must be in radians} \\ \textbf{Math.Pi} &\quad .... \quad \textit{is equal to } \pi = 3 \cdot 14129 \end{aligned}$$

ii The inverse trigonometrical functions available in $C\#$ are:

$$\begin{aligned} \textbf{Math.ASin}() &\quad .... \quad the\ result\ is\ an\ angle\ in\ radians\ from\ -\frac{\pi}{2}\ to\ \frac{\pi}{2} \\ \textbf{Math.ACos}() &\quad .... \quad the\ result\ is\ an\ angle\ in\ radians\ from\ 0\ to\ \pi \\ \textbf{Math.ATan}() &\quad .... \quad the\ result\ is\ an\ angle\ in\ radians\ from\ -\frac{\pi}{2}\ to\ \frac{\pi}{2} \end{aligned}$$

Example Consider the follow lines of code from $C\#$

$private\ void\ Form1_Paint(object\ sender, PaintEventArgs\ e)$

$\{$

$Graphics\ g = e.Graphics;$ $\quad //The\ graphics\ class$

$Penp = newPen(Color.Red);$ $\quad //The\ Pen\ class$

$double\ xValue = \mathbf{300} * \mathbf{Math.Cos(30} * \mathbf{Math.PI/180)};$

$double\ yValue = \mathbf{300} * \mathbf{Math.Sin(30} * \mathbf{Math.PI/180)};$

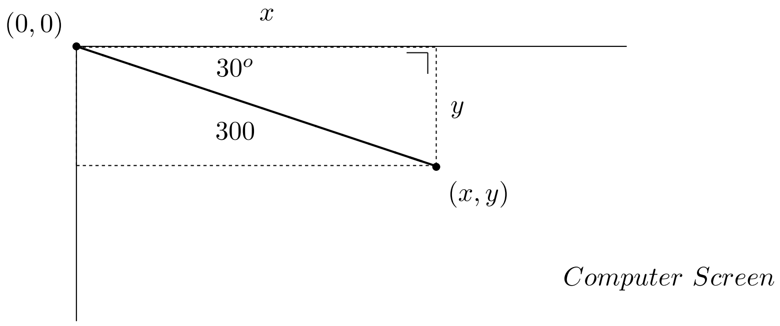
$g.DrawLine(p,\ 0,\ 0,\ (float)xValue,\ (float)yValue);$

$\}$

These lines of code will draw a line of length 300 at 30^o to the top of the screen starting at the origin i.e., the top left hand corner of the screen. The coordinates of the end-point of this line are given by the lines of code (including converting from degrees to radians)

$$double\ xValue = \mathbf{300 * Math.Cos(30 * Math.PI/180)};$$

$$double\ yValue = \mathbf{300 * Math.Sin(30 * Math.PI/180)};$$



From the diagram we have

$$\cos 30^o = \frac{x}{300}$$

$$\sin 30^o = \frac{y}{300}$$

Hence $x = 300\cos 30^o$ and $y = 300\sin 30^o$ and the coordinate of the end-point (x, y) of a line of length 300 at 30^o to the top of the screen starting at the origin is given as

$$\begin{aligned} (x, y) &= (300\cos 30^o, 300\sin 30^o) \\ &= (259\cdot 8, 150) \end{aligned}$$

These coordinates correspond to the lines of code (including converting from degrees to radians)

$$double\ xValue = \mathbf{300 * Math.Cos(30 * Math.PI/180)};$$

$$double\ yValue = \mathbf{300 * Math.Sin(30 * Math.PI/180)};$$

Exercise We have seen that the following lines of $C\#$ calculate the end-coordinates (x, y) of a line segment that starts at the origin $(0, 0)$ and makes an angle of 30^o to the top of the computer screen.

$$double\ xValue = \mathbf{300 * Math.Cos(30 * Math.PI/180)};$$

$$double\ yValue = \mathbf{300 * Math.Sin(30 * Math.PI/180)};$$

Write similar lines of code that calculate the end-coordinates of a line segment that starts (in both cases) at the origin and has

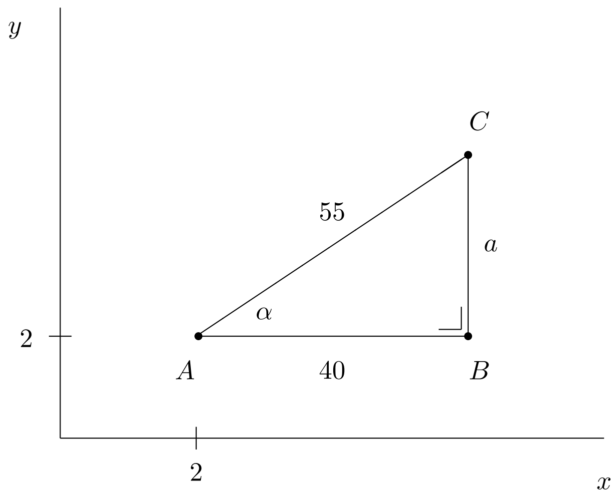
i magnitude 100 and makes an angle of 20^o to the top of the computer screen.

ii magnitude 175 and makes an angle of 40^o to the top of the computer screen.

In each case draw a coordinate diagram to illustrate your work.

Remark One of the most common applications of trigonometry is found in graphics programming when various shapes have to be drawn on the computer screen. The following example will illustrate these ideas.

Example Using basic trigonometry of right-angled triangles we can determine the coordinates of the two vertices B and C in the following triangle, given that A is at the point $(2, 2)$.



We have $A = (2, 2)$ and by observation $B = (42, 2)$. Using Pythagoras' theorem we have

$$\begin{aligned} a^2 + 40^2 &= 55^2 \\ a^2 &= 55^2 - 40^2 \\ a^2 &= 1,425 \\ a &= 37 \cdot 75 \end{aligned}$$

Hence $C = (42, 39 \cdot 75)$. **Alternatively**, we could write that

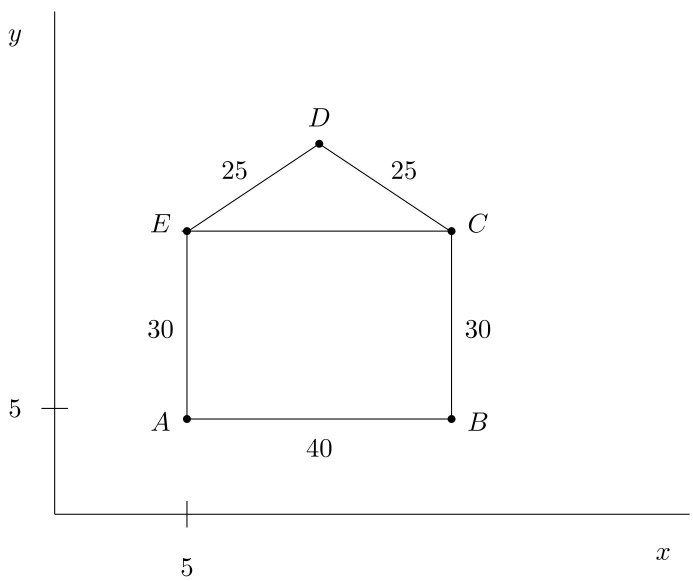
$$\cos\alpha = \frac{40}{55} = 0 \cdot 72727$$

So $\alpha = cos^{-1}(0 \cdot 72727) = 43 \cdot 3417^o$. Now

$$\begin{aligned} \sin\alpha &= \frac{a}{55} \\ a &= 55\sin\alpha \\ a &= 55\sin 43 \cdot 3417^o \\ a &= 37 \cdot 75 \end{aligned}$$

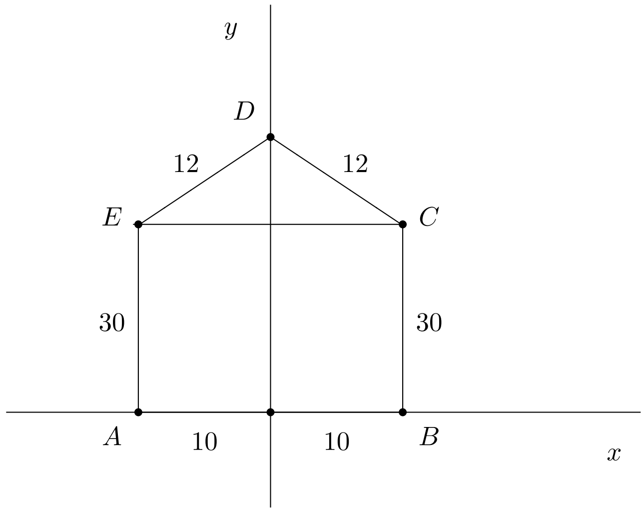
Hence $C = (42, 39 \cdot 75)$

Exercise Determine the coordinates of the points A, B, C, D and E in the following diagram, given that A is at the point $(5, 5)$.



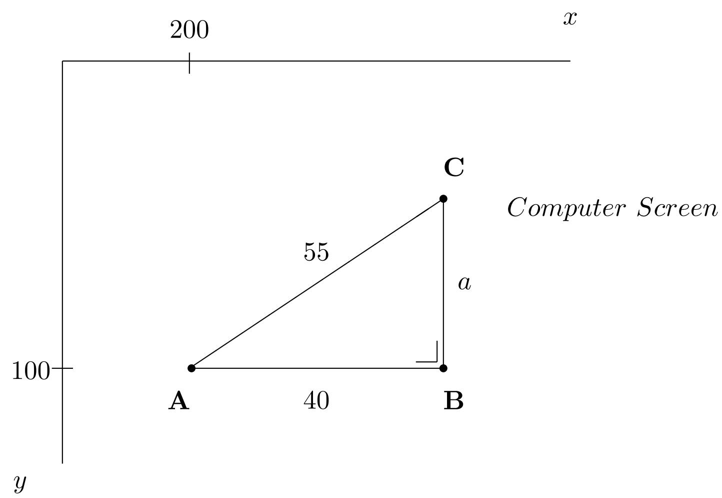
[**Solution**: *A=(5,5)* , *B=(45,5)*, *C=(45,35)* , *D=(25,50)*, *E=(5,35)*].

Exercise Determine the coordinates of the points A, B, C, D and E in the following diagram, given that the origin $(0, 0)$ is as indicated.



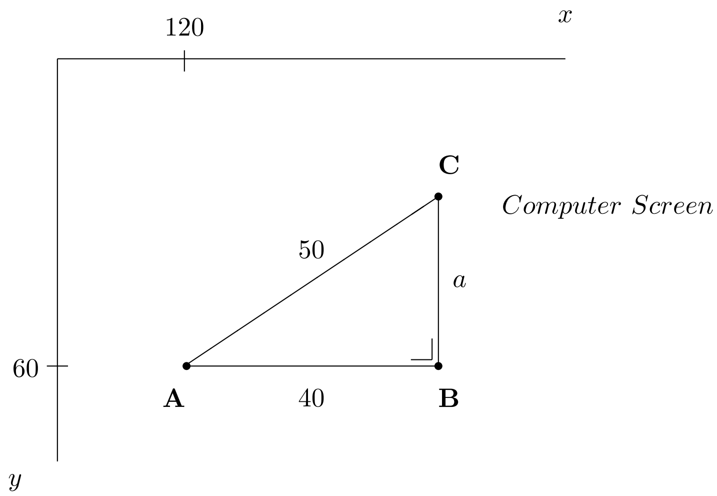
[**Solution**: *A=(-10,0)* , *B=(10,0)*, *C=(10,30)* , *D=(0,*$36 \cdot 63$*)*, *E=(-10,30)*].

Example The following *C#* code will draw this triangle with the bottom left corner at the point $(200, 100)$.



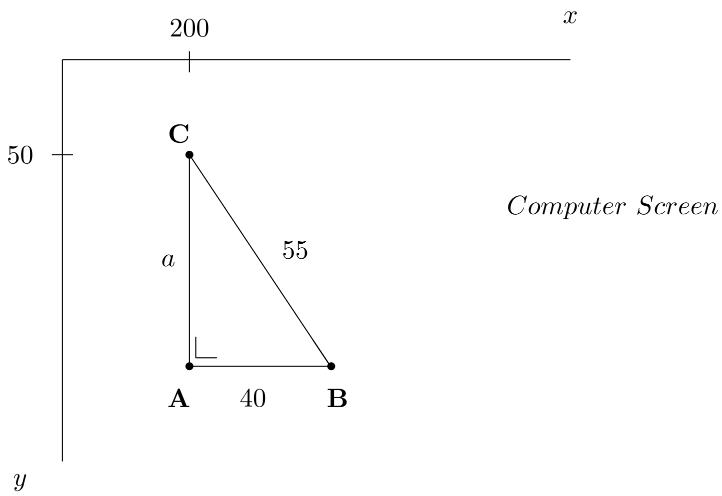
```
private void Form1_Paint(object sender, PaintEventArgse)
{
Graphicsg = e.Graphics; //The graphics class
Pen p = newPen(Color.Red); //class to change line colour etc.
double angle = Math.Acos(40.0/55.0);
double height = 55 * Math.Sin(angle);
g.DrawLine(p, 200, 100, 240, 100);                    // Drawline to draw line AB.
g.DrawLine(p, 240, 100, 240, 100 − (float)height);    // Drawline to draw line BC.
g.DrawLine(p, 240, 100 − (float)height, 200, 100);    // Drawline to draw line CA.
}
```

Exercise Complete the lines of $C\#$ code below to draw the triangle which has the bottom left corner at the point $(120, 60)$.




```
private void Form1_Paint(object sender, PaintEventArgse)
{
Graphicsg = e.Graphics; //The graphics class
Pen p = newPen(Color.Red); //class to change line colour etc.
double angle =
double height =
g.DrawLine(p, 120, 60, 160, 60);            // Drawline to draw line AB.
g.DrawLine(p,                        );     // Drawline to draw line BC.
g.DrawLine(p,                        );     // Drawline to draw line CA.
}
```

Exercise Complete the lines of $C\#$ code below to draw the triangle which has the top left corner at the point $(200, 50)$.



$private\ void\ Form1_Paint(object\ sender, PaintEventArgs e)$

{

$Graphics g = e.Graphics; //The\ graphics\ class$

$Pen\ p = new Pen(Color.Red); //class\ to\ change\ line\ colour\ etc.$

double angle =

double height =

g.**DrawLine**(**p**, **200**, **50**, **240**, **50**+(**float**)**height**); // *Drawline to draw line CB.*

g.**DrawLine**(**p**, **240**, **50** + (**float**)**height**, **200**, **50** + (**float**)**height**); // *Drawline to draw line BA.*

g.**DrawLine**(**p**,); // *Drawline to draw line AC.*

}