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## VECTORS IN THREE DIMENSIONS

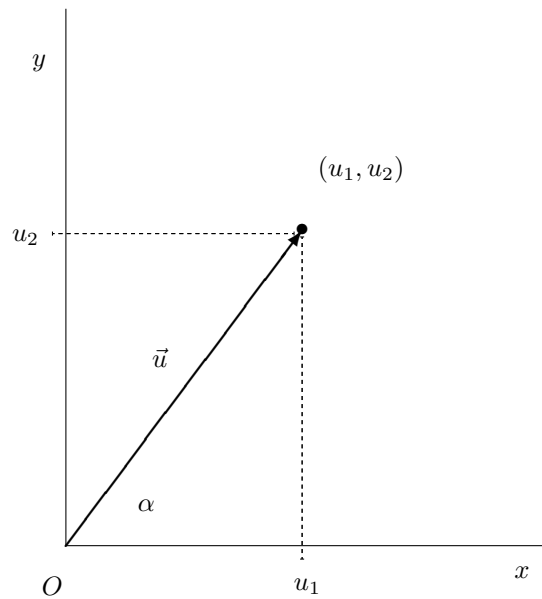
## 1 Vectors in Two Dimensions

A vector is an object which has *magnitude* and *direction*. Many physical quantities, such as velocity, acceleration, force, electric field and magnetic field are examples of vector quantities. Displacement between points may also be represented using vectors. We study some relationship between algebra and geometry. We shall first study some algebra which is motivated by geometric considerations. We then use the algebra later to better understand some problems in geometry. This mathematics will form the basis of the study of computer graphics. Vectors are central to the design of any two-dimensional or three-dimensional computer game. They are used to represent points in space, like corners of a door or window or the location of any object in a scene. They are also used to describe a direction, for example the orientation of a camera or the direction in which a gun is pointing.

A vector in two-dimensions  $\mathbb{R}^2$  can be described as an ordered pair  $\vec{u} = (u_1, u_2)$ , where  $u_1, u_2 \in \mathbb{R}$ .

**Definition** Two vectors  $\vec{u} = (u_1, u_2)$  and  $\vec{v} = (v_1, v_2)$  in  $\mathbb{R}^2$  are said to be equal, denoted by  $\vec{u} = \vec{v}$ , if and only if  $u_1 = v_1$  and  $u_2 = v_2$ .

Defining vectors in  $\mathbb{R}^2$  as ordered pairs of real numbers enables us to state precisely when two vectors are equal – it also provides us with the easiest way of defining addition and various kinds of multiplication. To describe the position of any point in two-dimensions we may choose two axes  $x$  and  $y$  which are mutually perpendicular and intersect in a point  $O$  called the origin, as shown.



Any point  $P$  in two dimensions corresponds the ordered pair  $(u_1, u_2)$  of real numbers, where  $u_1$  represents the magnitude of the component vector along x-axis and  $u_2$  represents the magnitude of the component vector along y-axis.

The magnitude of the vector  $\vec{u}$ , from Pythagoras' theorem, is given as

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2}$$

The direction of the vector  $\vec{u}$  is defined by  $\alpha$

$$\alpha = \tan^{-1}\left(\frac{u_2}{u_1}\right)$$

## 1.1 Addition of Vectors

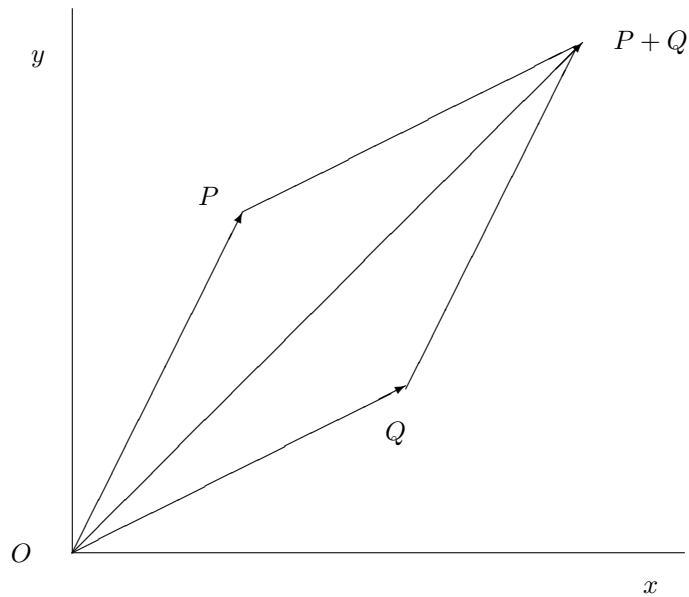
**Definition** For any two vectors  $\vec{u} = (u_1, u_2)$  and  $\vec{v} = (v_1, v_2)$  in  $\mathbb{R}^2$ , we define their sum to be

$$\begin{aligned}\vec{u} + \vec{v} &= (u_1, u_2) + (v_1, v_2) \\ &= (u_1 + v_1, u_2 + v_2)\end{aligned}$$

Similarly, we define their difference to be

$$\begin{aligned}\vec{u} - \vec{v} &= (u_1, u_2) - (v_1, v_2) \\ &= (u_1 - v_1, u_2 - v_2)\end{aligned}$$

Addition of vectors may be pictured using the 'parallelogram law'.



**Example** For the following pair of vectors  $\vec{u} = (1, 6)$  and  $\vec{v} = (-5, 2)$  in  $\mathbb{R}^2$ , we can calculate

$$\begin{aligned}\vec{u} + \vec{v} &= (1, 6) + (-5, 2) \\ &= (-4, 8) \\ \vec{u} - \vec{v} &= (1, 6) - (-5, 2) \\ &= (6, 4)\end{aligned}$$

**Theorem 1 (VECTOR ADDITION)**

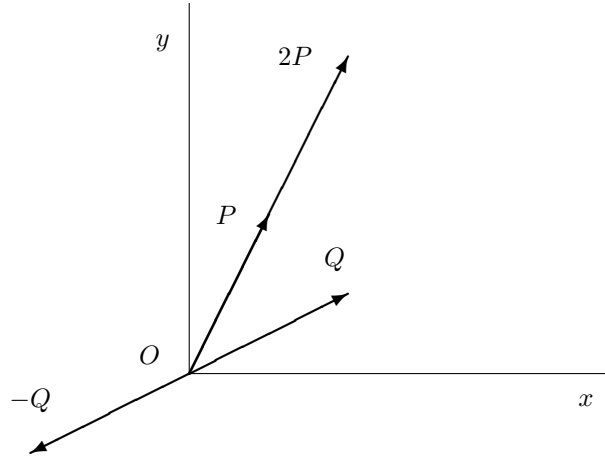
- i For every  $\vec{u}, \vec{v} \in \mathbb{R}^2$ , we have  $\vec{u} + \vec{v} \in \mathbb{R}^2$ .*
- ii For every  $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ , we have  $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ .*
- iii For every  $\vec{u} \in \mathbb{R}^2$ , we have  $\vec{u} + \vec{0} = \vec{u}$  where  $\vec{0} = (0, 0) \in \mathbb{R}^2$ .*
- iv For every  $\vec{u} \in \mathbb{R}^2$ , there exists  $\vec{v} \in \mathbb{R}^2$  such that  $\vec{u} + \vec{v} = \vec{0}$ .*
- v For every  $\vec{u}, \vec{v} \in \mathbb{R}^2$ , we have  $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ .*

## 1.2 Scalar Multiplication of Vectors

**Definition** For any vector  $\vec{u} = (u_1, u_2)$  in  $\mathbb{R}^2$  and any scalar  $c \in \mathbb{R}$ , we define the scalar multiple to be

$$c\vec{u} = c(u_1, u_2) = (cu_1, cu_2)$$

Scalar multiplication may be pictured as follows –



**Example** For the following pair of vectors  $\vec{u} = (2, 3)$  and  $\vec{v} = (-1, 5)$  in  $\mathbb{R}^2$ , we can evaluate

$$\begin{aligned} 2\vec{u} + 4\vec{v} &= 2(2, 3) + 4(-1, 5) \\ &= (4, 6) + (-4, 20) \\ &= (0, 26) \\ 2\vec{u} - \vec{v} &= 2(2, 3) - (-1, 5) \\ &= (4, 6) - (-1, 5) \\ &= (5, 1) \end{aligned}$$

**Theorem 2 (SCALAR MULTIPLICATION)**

- i For every  $c \in \mathbb{R}$  and  $\vec{u} \in \mathbb{R}^2$ , we have  $c\vec{u} \in \mathbb{R}^2$ .*
- ii For every  $c \in \mathbb{R}$  and  $\vec{u}, \vec{v} \in \mathbb{R}^2$ , we have  $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ .*
- iii For every  $a, b \in \mathbb{R}$  and  $\vec{u} \in \mathbb{R}^2$ , we have  $(a + b)\vec{u} = a\vec{u} + b\vec{u}$ .*
- iv For every  $a, b \in \mathbb{R}$  and  $\vec{u} \in \mathbb{R}^2$ , we have  $(ab)\vec{u} = a(b\vec{u})$ .*
- v For every  $\vec{u} \in \mathbb{R}^2$ , we have  $1\vec{u} = \vec{u}$ .*

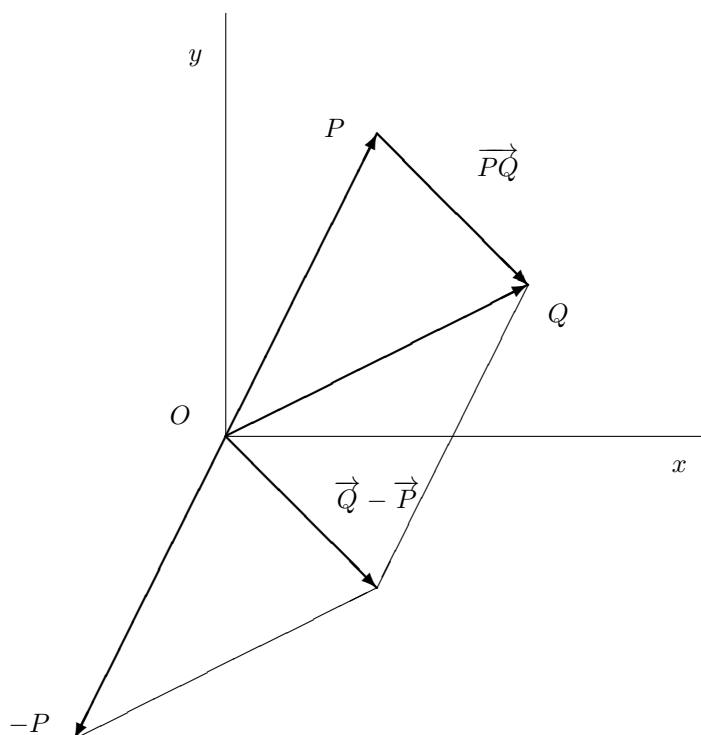
**Exercise** For the following pair of vectors  $\vec{u} = (4, -3)$  and  $\vec{v} = (1, 7)$  in  $\mathbb{R}^2$ , evaluate

- i  $3\vec{u} + 3\vec{v}$ ,
- ii  $\vec{u} + 2\vec{v}$ ,
- iii  $7\vec{u} - 3\vec{v}$ .

**Remark** There is another way in which vectors may be pictured – namely as ‘arrows’ in two dimensions. The vector  $(u_1, u_2)$  can be pictured by an arrow with initial point  $O$  and terminal point  $(u_1, u_2)$ . It is, however, convenient to picture vectors in a more general way. Consider an arrow with the initial point  $P = (x_1, y_1)$  and terminal point  $Q = (x_2, y_2)$ . This arrow is denoted by  $\overrightarrow{PQ}$ . We define

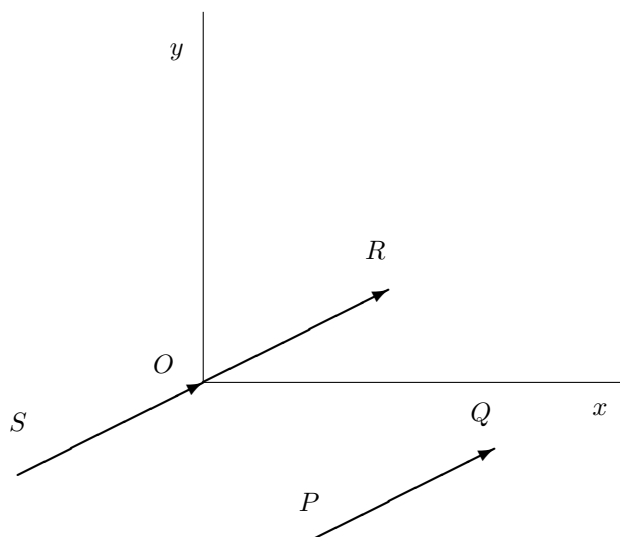
$$\begin{aligned}\overrightarrow{PQ} &= (x_2 - x_1, y_2 - y_1) \\ &= \vec{Q} - \vec{P}\end{aligned}$$

We can picture this as follows – using the parallelogram law



Note that this vector is already represented by the arrow with initial point  $O$  and terminal point  $(x_2 - x_1, y_2 - y_1)$ . In fact, any one vector may be represented by infinitely many arrows. We define two arrows to be *equivalent* whenever their corresponding components are equal – it is the components, and not the individual initial and terminal points, which are used to see if two arrows are equivalent. Since components are determined by the length and direction of an arrow we can state that two arrows are equivalent whenever they have the same length and direction. Since one vector is now represented by any one of infinitely many equivalent arrows, we agree to regard these equivalent arrows as equal. The end result is that we may picture a vector as an arrow which has a given length and lies in a given direction, and may be positioned between any pair of points provided that the points determine the same length and direction.

**Example** Let  $P = (2, -5)$ ,  $Q = (3, -2)$ ,  $R = (1, 3)$  and  $S = (-1, -3)$  be four points in two dimensions, as shown.



Then

$$\overrightarrow{PQ} = (1, 3)$$

$$\overrightarrow{OR} = (1, 3)$$

$$\overrightarrow{SO} = (1, 3)$$

These arrows represent the same vector, namely,  $A = (1, 3)$  and we write

$$\overrightarrow{PQ} = \overrightarrow{OR} = \overrightarrow{SO} = A = (1, 3)$$

**Exercise** In each case write the vector  $\vec{u}$  in terms of components

- i  $\vec{u}$  is a vector from the point  $A(2, -5)$  to the point  $B(0, 4)$ ,
- ii  $\vec{u}$  is a vector from the point  $A(-1, -3)$  to the point  $B(5, 2)$ ,
- iii  $\vec{u}$  is a vector from the point  $A(5, 12)$  to the point  $B(-3, -6)$ .

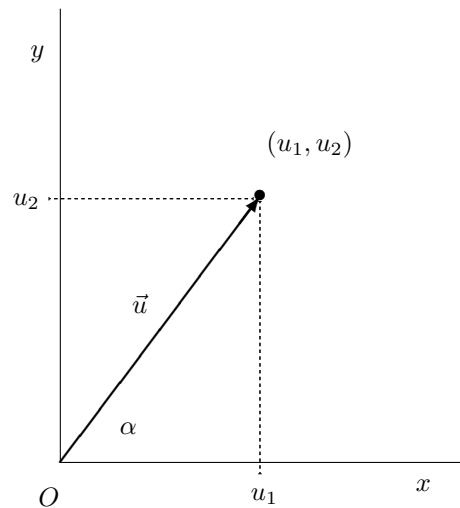
### 1.3 Magnitude and Direction

**Definition** For any vector  $\vec{u} = (u_1, u_2)$  in  $\mathbb{R}^2$ , we define the magnitude of  $\vec{u}$  to be the non-negative real number

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2}$$

The direction of the vector  $\vec{u}$  is defined by  $\alpha$

$$\alpha = \tan^{-1}\left(\frac{u_2}{u_1}\right)$$



#### Remark

- i Suppose that  $P = (x_1, y_1)$  and  $Q = (x_2, y_2)$  are two points in  $\mathbb{R}^2$ . To calculate the distance  $d(P, Q)$  between the two points, we must first find a vector from  $P$  to  $Q$ . This is given by  $(x_2 - x_1, y_2 - y_1)$ . The distance  $d(P, Q)$  is then the magnitude of this vector, so that

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Hence, the definition of the magnitude (or norm) of a vector  $\vec{u}$  is simply the distance from  $O$  to the point  $(u_1, u_2)$ .

- ii A vector of magnitude 1 is called a *unit vector* or *normalized vector*. Any non-zero vector  $\vec{u}$  determines a unit vector

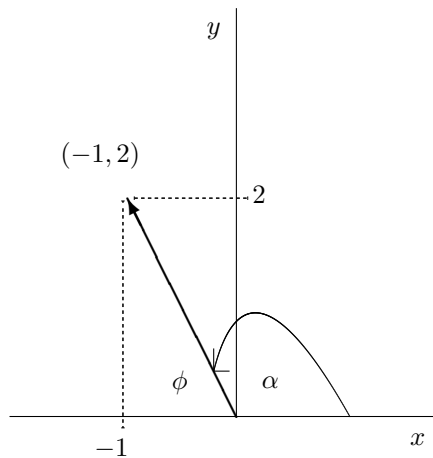
$$\frac{1}{\|\vec{u}\|}\vec{u} = \left(\frac{u_1}{\|\vec{u}\|}, \frac{u_2}{\|\vec{u}\|}\right)$$

**Example** The vector  $\vec{u} = (3, 4)$  has magnitude 5. This vector has direction  $\alpha = 53 \cdot 13^\circ$ .

**Example** The vector  $\vec{u} = (2, 5)$  has magnitude  $5 \cdot 385$ . This vector has direction  $\alpha = 68 \cdot 199^\circ$ .

**Example** The vector  $\vec{u} = (1, 2)$  has magnitude  $2 \cdot 236$ . This vector has direction  $\alpha = 63 \cdot 434^\circ$ .

**Example** The vector  $\vec{u} = (-1, 2)$  has magnitude  $2 \cdot 236$ .



From the diagram

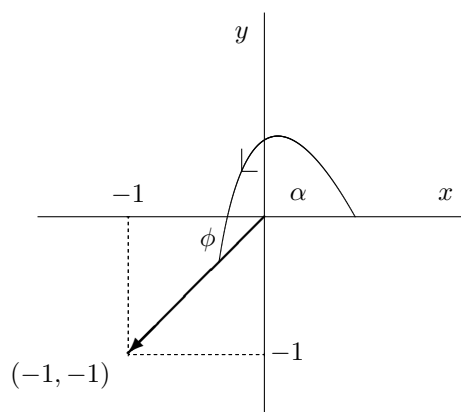
$$\phi = \tan^{-1}\left(\frac{2}{1}\right) = 63 \cdot 43^\circ$$

Hence  $\alpha = 180^\circ - 63 \cdot 43^\circ = 116 \cdot 57^\circ$ .

**Note** The angle of direction  $\alpha$  is the always quoted relative to the **positive x-axis**.

**Example** The vector  $\vec{u} = (-2, 3)$  has magnitude  $3 \cdot 606$ . This vector has direction  $\alpha = 123 \cdot 69^\circ$ .

**Example** The vector  $\vec{u} = (-1, -1)$  has magnitude  $1 \cdot 414$ .





From the diagram

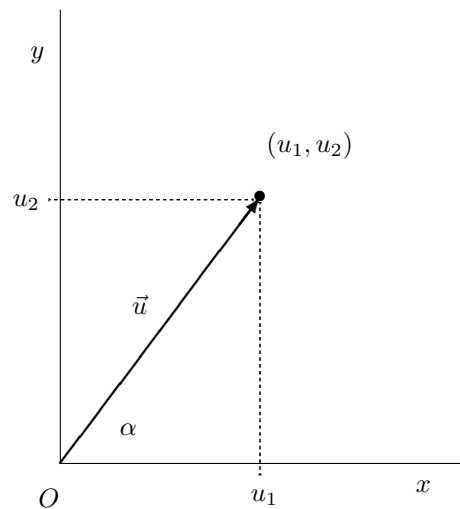
$$\phi = \tan^{-1}\left(\frac{1}{1}\right) = 45^\circ$$

Hence  $\alpha = 180^\circ + 45^\circ = 225^\circ$ .

**Example** The vector  $\vec{u} = (-4, -5)$  has magnitude  $6 \cdot 403$ . This vector has direction  $\alpha = 231 \cdot 34^\circ$ .

**Example** The vector  $\vec{u} = (-9, -12)$  has magnitude 15. This vector has direction  $\alpha = 233 \cdot 13^\circ$ .

**Remark** We may be required to determine the components  $u_1$  and  $u_2$  of the vector  $\vec{u} = (u_1, u_2)$  when presented with the magnitude and direction of  $\vec{u}$  only. If, for example, the magnitude and direction of two distinct vectors are presented – converting each vector to components will allow for simpler addition (subtraction or scalar multiplication) of the vectors.



For  $\vec{u} = (u_1, u_2)$  we can write,

$$u_1 = \|\vec{u}\| \cos \alpha$$

$$u_2 = \|\vec{u}\| \sin \alpha$$

**Example** The vector  $\vec{u}$  with magnitude 5 and direction  $\alpha = 40^\circ$  with positive x-axis has components

$$u_1 = 5 \cos 40^\circ$$

$$u_2 = 5 \sin 40^\circ$$

Hence  $\vec{u} = (3 \cdot 83, 3 \cdot 21)$ .

**Example** The vector  $\vec{u}$  with magnitude 200 and direction  $\alpha = 210^\circ$  with positive x-axis has components

$$u_1 = 200 \cos 210^\circ$$

$$u_2 = 200 \sin 210^\circ$$

Hence  $\vec{u} = (-173.21, -100)$ .

**Example** The vector  $\vec{u}$  has magnitude 10 and direction  $\alpha = 45^\circ$  with positive x-axis. The vector  $\vec{v}$  has magnitude 15 and direction  $\alpha = 205^\circ$  with positive x-axis.

Determine the magnitude and direction of each of the following vectors:

i  $\vec{u} + \vec{v}$

ii  $\vec{u} - 2\vec{v}$

iii  $2\vec{u} - 3\vec{v}$

**Solution:** Let  $\vec{u} = (u_1, u_2)$  and  $\vec{v} = (v_1, v_2)$ . Now

$$u_1 = 10 \cos 45^\circ = 7.07$$

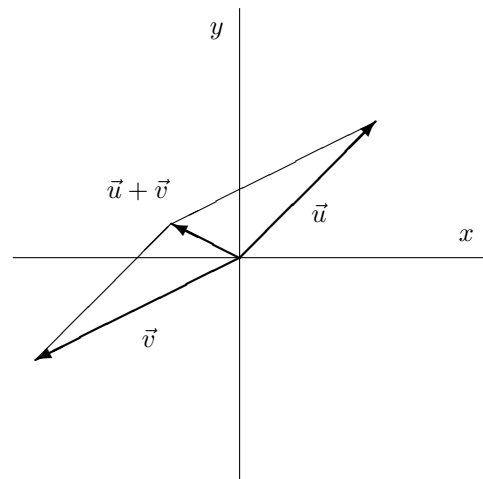
$$u_2 = 10 \sin 45^\circ = 7.07$$

$$v_1 = 15 \cos 205^\circ = -13.59$$

$$v_2 = 15 \sin 205^\circ = -6.34$$

Hence  $\vec{u} = (7.07, 7.07)$  and  $\vec{v} = (-13.59, -6.34)$ .

i



$$\begin{aligned}
\vec{u} + \vec{v} &= (7 \cdot 07, 7 \cdot 07) + (-13 \cdot 59, -6 \cdot 34) \\
&= (7 \cdot 07 - 13 \cdot 59, 7 \cdot 07 - 6 \cdot 34) \\
&= (-6 \cdot 52, 0 \cdot 73)
\end{aligned}$$

The vector  $\vec{u} + \vec{v}$  has magnitude  $6 \cdot 56$ . This vector has direction  $\alpha = 96 \cdot 39^\circ$ .

ii

$$\begin{aligned}
\vec{u} - 2\vec{v} &= (7 \cdot 07, 7 \cdot 07) - 2(-13 \cdot 59, -6 \cdot 34) \\
&= (7 \cdot 07, 7 \cdot 07) - (-27 \cdot 18, -12 \cdot 68) \\
&= (7 \cdot 07 + 27 \cdot 18, 7 \cdot 07 + 12 \cdot 68) \\
&= (34 \cdot 25, 19 \cdot 75)
\end{aligned}$$

The vector  $\vec{u} - 2\vec{v}$  has magnitude  $39 \cdot 54$ . This vector has direction  $\alpha = 29 \cdot 97^\circ$ .

iii

$$\begin{aligned}
2\vec{u} - 3\vec{v} &= 2(7 \cdot 07, 7 \cdot 07) - 3(-13 \cdot 59, -6 \cdot 34) \\
&= (14 \cdot 14, 14 \cdot 14) - (-40 \cdot 77, -19 \cdot 02) \\
&= (14 \cdot 14 + 40 \cdot 77, 14 \cdot 14 + 19 \cdot 02) \\
&= (54 \cdot 91, 33 \cdot 16)
\end{aligned}$$

The vector  $2\vec{u} - 3\vec{v}$  has magnitude  $64 \cdot 15$ . This vector has direction  $\alpha = 31 \cdot 13^\circ$ .

**Remark** For computer games and graphics programming the component representation of a vector with round (or square) brackets is used. This notation will be used in all modern computer programming languages. In *C#*, for example, the **DrawLine()** method draws a line from one vector point  $(x1, y1)$  to a second vector point  $(x2, y2)$  on the graphics form:

**g.DrawLine(p, x1, y1, x2, y2); //Drawline method**

The following lines of code from *C#* will draw a line from the point (100, 150) to (300, 400):

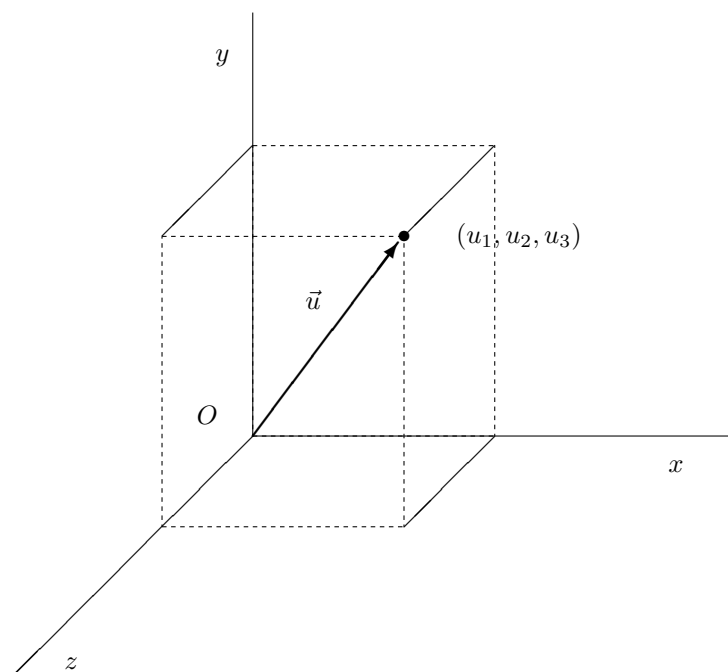
```
private void Form1_Paint(object sender, PaintEventArgs e)
{
    Graphics g = e.Graphics; //The graphics class
    Pen p = new Pen(Color.Red); //The Pen class
    g.DrawLine(p, 100, 150, 300, 400); //Drawline method
}
```

## 2 Vectors in Three Dimensions

A vector in three-dimensions  $\mathbb{R}^3$  can be described as an ordered triple  $\vec{u} = (u_1, u_2, u_3)$ , where  $u_1, u_2, u_3 \in \mathbb{R}$ .

**Definition** Two vectors  $\vec{u} = (u_1, u_2, u_3)$  and  $\vec{v} = (v_1, v_2, v_3)$  in  $\mathbb{R}^3$  are said to be equal, denoted by  $\vec{u} = \vec{v}$ , if and only if,  $u_1 = v_1$ ,  $u_2 = v_2$  and  $u_3 = v_3$ .

Defining vectors in  $\mathbb{R}^3$  as ordered triples of real numbers enables us to state precisely when two vectors are equal – it also provides us with the easiest way of defining addition and various kinds of multiplication, as we will show later. To describe the position of any point in space, we may choose three axes  $x, y$  and  $z$  which are mutually perpendicular and intersect in a point  $O$  called the origin, as shown.



Any point in space corresponds to the ordered triple  $(u_1, u_2, u_3)$  of real numbers, where  $u_1$  represents the magnitude of the component vector along  $x$ -axis,  $u_2$  represents the magnitude of the component vector along  $y$ -axis and  $u_3$  represents the magnitude of the component vector along the  $z$ -axis.

The magnitude of the vector  $\vec{u}$ , from Pythagoras' theorem, is given as

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

The direction of the vector in three-dimensions  $\mathbb{R}^3$  is defined by three angles  $\theta_x$ ,  $\theta_y$  and  $\theta_z$  the vector makes with the x-axis, y-axis and z-axis respectively. For a vector  $\vec{u} = (u_1, u_2, u_3)$  in  $\mathbb{R}^3$

$$u_1 = \|\vec{u}\| \cos \theta_x$$

$$u_2 = \|\vec{u}\| \cos \theta_y$$

$$u_3 = \|\vec{u}\| \cos \theta_z$$

## 2.1 Addition of Vectors

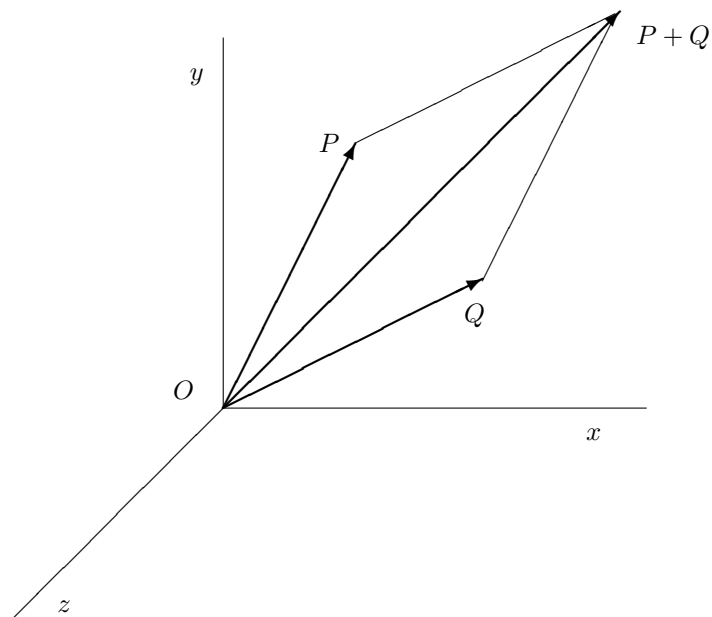
**Definition** For any two vectors  $\vec{u} = (u_1, u_2, u_3)$  and  $\vec{v} = (v_1, v_2, v_3)$  in  $\mathbb{R}^3$ , we define their sum to be

$$\begin{aligned} \vec{u} + \vec{v} &= (u_1, u_2, u_3) + (v_1, v_2, v_3) \\ &= (u_1 + v_1, u_2 + v_2, u_3 + v_3) \end{aligned}$$

Similarly, we define their difference to be

$$\begin{aligned} \vec{u} - \vec{v} &= (u_1, u_2, u_3) - (v_1, v_2, v_3) \\ &= (u_1 - v_1, u_2 - v_2, u_3 - v_3) \end{aligned}$$

Addition of vectors may be pictured using the ‘parallelogram law’.



**Example** For the following pair of vectors  $\vec{u} = (8, 4, -3)$  and  $\vec{v} = (-2, 2, 0)$  in  $\mathbb{R}^3$ , we can calculate, for example

$$\begin{aligned}\vec{u} + \vec{v} &= (8, 4, -3) + (-2, 2, 0) \\ &= (6, 6, -3) \\ \vec{u} - \vec{v} &= (8, 4, -3) - (-2, 2, 0) \\ &= (10, 2, -3)\end{aligned}$$

**Theorem 3 (VECTOR ADDITION)**

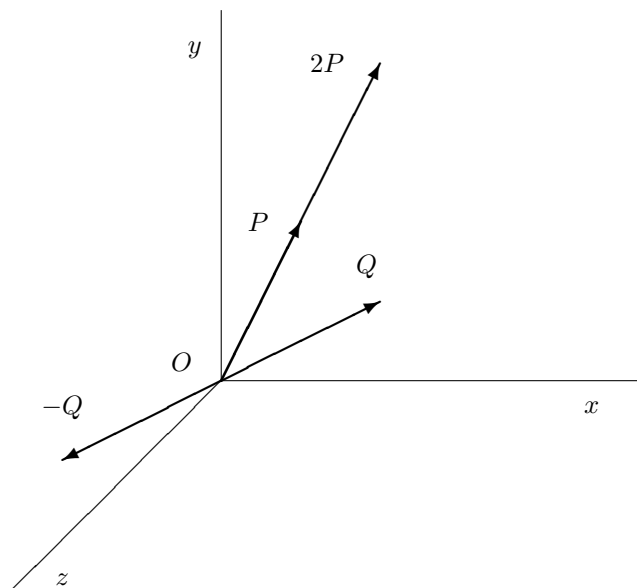
- i* For every  $\vec{u}, \vec{v} \in \mathbb{R}^3$ , we have  $\vec{u} + \vec{v} \in \mathbb{R}^3$ .
- ii* For every  $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ , we have  $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ .
- iii* For every  $\vec{u} \in \mathbb{R}^3$ , we have  $\vec{u} + \vec{0} = \vec{u}$  where  $\vec{0} = (0, 0, 0) \in \mathbb{R}^3$ .
- iv* For every  $\vec{u} \in \mathbb{R}^3$ , there exists  $\vec{v} \in \mathbb{R}^3$  such that  $\vec{u} + \vec{v} = \vec{0}$ .
- v* For every  $\vec{u}, \vec{v} \in \mathbb{R}^3$ , we have  $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ .

## 2.2 Scalar Multiplication of Vectors

**Definition** For any vector  $\vec{u} = (u_1, u_2, u_3)$  in  $\mathbb{R}^3$  and any scalar  $c \in \mathbb{R}$ , we define the scalar multiple to be

$$c\vec{u} = c(u_1, u_2, u_3) = (cu_1, cu_2, cu_3)$$

Scalar multiplication may be pictured as follows –



**Example** For the following pair of vectors  $\vec{u} = (2, 3, -4)$  and  $\vec{v} = (-1, 3, 8)$  in  $\mathbb{R}^3$ , we can evaluate, for example

$$\begin{aligned} 2\vec{u} + 4\vec{v} &= 2(2, 3, -4) + 4(-1, 3, 8) \\ &= (4, 6, -8) + (-4, 12, 32) \\ &= (0, 18, 24) \\ 2\vec{u} - \vec{v} &= 2(2, 3, -4) - (-1, 3, 8) \\ &= (4, 6, -8) - (-1, 3, 8) \\ &= (5, 3, -16) \end{aligned}$$

**Theorem 4 (SCALAR MULTIPLICATION)**

- i For every  $c \in \mathbb{R}$  and  $\vec{u} \in \mathbb{R}^3$ , we have  $c\vec{u} \in \mathbb{R}^3$ .*
- ii For every  $c \in \mathbb{R}$  and  $\vec{u}, \vec{v} \in \mathbb{R}^3$ , we have  $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ .*
- iii For every  $a, b \in \mathbb{R}$  and  $\vec{u} \in \mathbb{R}^3$ , we have  $(a + b)\vec{u} = a\vec{u} + b\vec{u}$ .*
- iv For every  $a, b \in \mathbb{R}$  and  $\vec{u} \in \mathbb{R}^3$ , we have  $(ab)\vec{u} = a(b\vec{u})$ .*
- v For every  $\vec{u} \in \mathbb{R}^3$ , we have  $1\vec{u} = \vec{u}$ .*

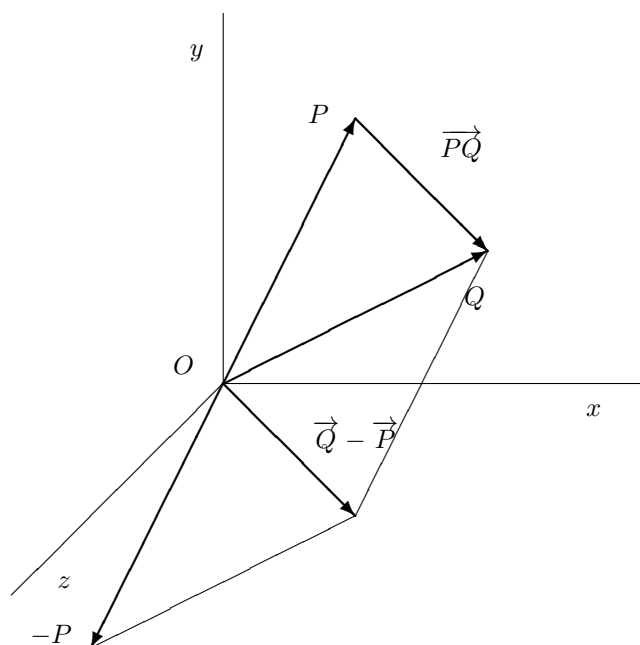
**Exercise** For the following pair of vectors  $\vec{u} = (1, -3, 5)$  and  $\vec{v} = (1, -2, 4)$  in  $\mathbb{R}^3$ , evaluate

- i  $2\vec{u} + 3\vec{v}$ ,
- ii  $\vec{u} - 5\vec{v}$ ,
- iii  $4\vec{u} + 3\vec{v}$ .

**Remark** There is another way in which vectors may be pictured – namely as ‘arrows’ in three dimensions. The vector  $(u_1, u_2, u_3)$  can be pictured by an arrow with initial point  $O$  and terminal point  $(u_1, u_2, u_3)$ . It is, however, convenient to picture vectors in a more general way. Consider an arrow with the initial point  $P = (x_1, y_1, z_1)$  and terminal point  $Q = (x_2, y_2, z_2)$ . This arrow is denoted by  $\overrightarrow{PQ}$ . We define

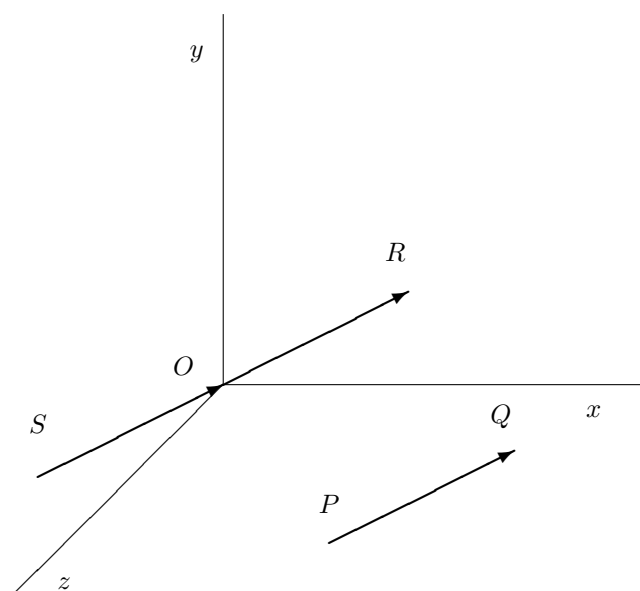
$$\begin{aligned} \overrightarrow{PQ} &= (x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ &= \vec{Q} - \vec{P} \end{aligned}$$

We can picture this as follows – using the parallelogram law



Recall that we may picture a vector as an arrow which has a given length and lies in a given direction, and may be positioned between any pair of points provided that the points determine the same length and direction. The following example illustrates this point.

**Example** Let  $P = (2, -5, 4)$ ,  $Q = (3, -2, 6)$ ,  $R = (1, 3, 2)$  and  $S = (-1, -3, -2)$  be four points in space, as shown.



Then

$$\overrightarrow{PQ} = (1, 3, 2)$$

$$\overrightarrow{OR} = (1, 3, 2)$$

$$\overrightarrow{SO} = (1, 3, 2)$$



These arrows represent the same vector, namely,  $A = (1, 3, 2)$  and we write

$$\overrightarrow{PQ} = \overrightarrow{OR} = \overrightarrow{SO} = A = (1, 3, 2)$$

**Exercise** In each case write the vector  $\vec{u}$  in terms of components

- i  $\vec{u}$  is a vector from the point  $A(1, -5, 4)$  to the point  $B(2, 0, 4)$ ,
- ii  $\vec{u}$  is a vector from the point  $A(1, 2, 3)$  to the point  $B(4, 5, 6)$ ,
- iii  $\vec{u}$  is a vector from the point  $A(-2, 1, 9)$  to the point  $B(3, -6, 8)$ .

[**Solution:**  $\vec{u} = (1, 5, 0)$  ,  $\vec{u} = (3, 3, 3)$  ,  $\vec{u} = (5, -7, -1)$ ].

**Exercise** Let

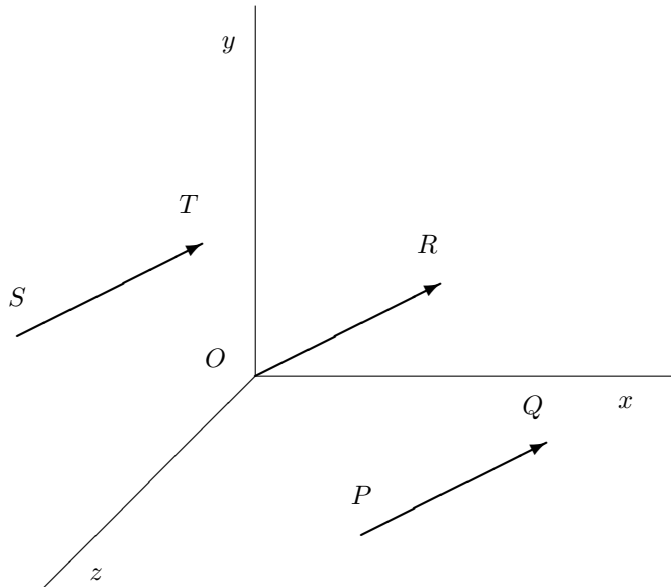
$$\begin{aligned}\vec{u} &= (2, -1, 3) \\ \vec{v} &= (-4, 2 \cdot 5, 3) \\ \vec{w} &= (1, 1, -2)\end{aligned}$$

Write each of the following in terms of components

- i  $2\vec{u}$
- ii  $3\vec{u} - 2\vec{v}$
- iii  $\vec{v} - 2\vec{u} + 4\vec{w}$
- iv  $2(\vec{u} + \vec{v}) - \vec{w}$

[**Solution:** i  $(4, -2, 6)$  , ii  $(14, -8, 3)$  , iii  $(-4, 8 \cdot 5, -11)$  , iv  $(-5, 2, 14)$  ].

**Exercise** Let  $P = (2, -6, 8)$ ,  $Q = (6, -2, 5)$ ,  $R = (4, 4, -3)$ ,  $S = (-10, 2, 5)$  and  $T = (-6, 6, 2)$  be five points in space, as shown.



Show that  $\overrightarrow{PQ} = \overrightarrow{OR} = \overrightarrow{ST}$ .

### 2.3 Magnitude and Direction

**Definition** For any vector  $\vec{u} = (u_1, u_2, u_3)$  in  $\mathbb{R}^3$ , we define the magnitude of  $\vec{u}$  to be the non-negative real number

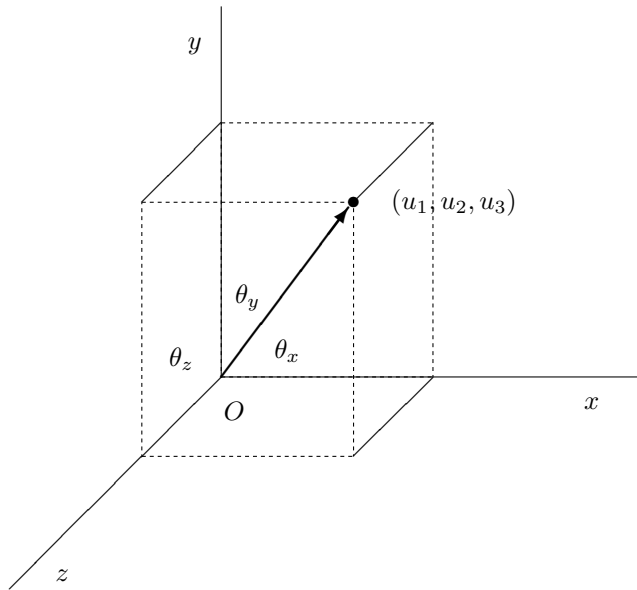
$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

The direction of the vector in three-dimensions  $\mathbb{R}^3$  is defined by three angles  $\theta_x$ ,  $\theta_y$  and  $\theta_z$  the vector makes with the x-axis, y-axis and z-axis respectively. For a vector  $\vec{u} = (u_1, u_2, u_3)$  in  $\mathbb{R}^3$

$$\cos \theta_x = \frac{u_1}{\|\vec{u}\|}$$

$$\cos \theta_y = \frac{u_2}{\|\vec{u}\|}$$

$$\cos \theta_z = \frac{u_3}{\|\vec{u}\|}$$



These angles are difficult to picture, since they are not in the  $x$ -plane only or the  $y$ -plane only or the  $z$ -plane only. For  $\vec{u} = (u_1, u_2, u_3)$  in  $\mathbb{R}^3$ , we can write,

$$\begin{aligned} u_1 &= \|\vec{u}\| \cos \theta_x \\ u_2 &= \|\vec{u}\| \cos \theta_y \\ u_3 &= \|\vec{u}\| \cos \theta_z \end{aligned}$$

These equations will determine the components  $u_1$ ,  $u_2$  and  $u_3$  of the vector  $\vec{u} = (u_1, u_2, u_3)$  in  $\mathbb{R}^3$  when presented with the magnitude and direction of  $\vec{u}$  only.

Furthermore

$$\cos^2 \theta_x + \cos^2 \theta_y + \cos^2 \theta_z = 1$$

**Remark** A vector of magnitude 1 is called a *unit vector* or *normalised vector*. Any non-zero vector determines a unit vector

$$\begin{aligned} \frac{1}{\|\vec{u}\|} \vec{u} &= \left( \frac{u_1}{\|\vec{u}\|}, \frac{u_2}{\|\vec{u}\|}, \frac{u_3}{\|\vec{u}\|} \right) \\ \frac{1}{\|\vec{u}\|} \vec{u} &= (\cos \theta_x, \cos \theta_y, \cos \theta_z) \end{aligned}$$

This normalised form may be used to conveniently calculate the angles  $\theta_x, \theta_y, \theta_z$  for a given vector  $\vec{u} = (u_1, u_2, u_3)$  in  $\mathbb{R}^3$ . This form is usually used to describe the direction of any vector in  $\mathbb{R}^3$ .

**Example** We can write  $\vec{v} = (2, 3, 4)$  as a unit vector (or normalised vector) along  $\vec{u}$

$$\frac{1}{\|\vec{u}\|} \vec{u} = \left( \frac{u_1}{\|\vec{u}\|}, \frac{u_2}{\|\vec{u}\|}, \frac{u_3}{\|\vec{u}\|} \right)$$

Now  $\|\vec{u}\| = \sqrt{29}$ . Hence

$$\frac{1}{\|\vec{u}\|} \vec{u} = \left( \frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}} \right) = (\cos \theta_x, \cos \theta_y, \cos \theta_z)$$

It follows that

$$\begin{aligned} \cos \theta_x &= \left( \frac{2}{\sqrt{29}} \right) = 0.3714 \\ \cos \theta_y &= \left( \frac{3}{\sqrt{29}} \right) = 0.5571 \\ \cos \theta_z &= \left( \frac{4}{\sqrt{29}} \right) = 0.7428 \end{aligned}$$

Hence

$$\begin{aligned} \theta_x &= 68.20^\circ \\ \theta_y &= 56.15^\circ \\ \theta_z &= 42.03^\circ \end{aligned}$$

**Exercise** For each of the following vectors in  $\mathbb{R}^3$ , find the angles  $\theta_x, \theta_y, \theta_z$ , i.e., the angle the vector makes with the x-axis, y-axis and z-axis respectively.

i  $\vec{u} = (-2, 1, 1)$

ii  $\vec{u} = (1, -1, 1)$

iii  $\vec{u} = (-4, -2, 2)$

[**Solution:** i  $\theta_x = 144.74^\circ$  ,  $\theta_y = 65.91^\circ$  ,  $\theta_z = 65.91^\circ$   
 ii  $\theta_x = 54.74^\circ$  ,  $\theta_y = 125.26^\circ$  ,  $\theta_z = 54.74^\circ$   
 iii  $\theta_x = 144.74^\circ$  ,  $\theta_y = 114.09^\circ$  ,  $\theta_z = 65.91^\circ$  ].

**Remark** A very common problem in games is that an object moves a distance in a particular direction and we have to determine where it ends up, so that we can draw it again in the new position. Suppose that an object is at position  $A(1, 2, 3)$  in one frame and it moves 10 units in the direction  $\theta_x = 75^\circ$ ,  $\theta_y = 50^\circ$  and  $\theta_z = 43.86^\circ$  before the next frame. What is the new position in the second frame?

Let  $\vec{u} = (u_1, u_2, u_3)$  with,

$$\begin{aligned} u_1 &= \|\vec{u}\| \cos \theta_x = 10 \cos 75^\circ \\ u_2 &= \|\vec{u}\| \cos \theta_y = 10 \cos 50^\circ \\ u_3 &= \|\vec{u}\| \cos \theta_z = 10 \cos 43 \cdot 86^\circ \end{aligned}$$

Hence  $\vec{u} = (2 \cdot 588, 6 \cdot 43, 7 \cdot 21)$ . The new position in the second frame will be

$$\begin{aligned} A + \vec{u} &= (1, 2, 3) + (2 \cdot 588, 6 \cdot 43, 7 \cdot 21) \\ &= (3 \cdot 588, 8 \cdot 43, 10 \cdot 21) \end{aligned}$$

**Exercise** For each of the following objects defined in frame one by a vector  $A$ , find its new position in frame two:

- i The object starts from  $A(2, 3, 1)$  and moves 7 units in the direction  $\theta_x = 90^\circ$ ,  $\theta_y = 130^\circ$  and  $\theta_z = 40^\circ$ .
- ii The object starts from  $A(1, 2, 3)$  and moves  $2 \cdot 8$  units in the direction  $\theta_x = 120^\circ$ ,  $\theta_y = 60^\circ$ .
- iii The object starts from  $A(-1, 1, 2)$  and moves  $1 \cdot 5$  units in the direction  $\theta_x = 50^\circ$ ,  $\theta_y = 70^\circ$ .

**[Solution:** i  $(2, -1 \cdot 50, 6 \cdot 36)$   
 ii  $(-0 \cdot 4, 3 \cdot 4, 4 \cdot 98)$   
 iii  $(0 \cdot 03, 1 \cdot 96, 2 \cdot 51)$  ].

## 2.4 The Scalar Product

**Definition** Suppose that  $\vec{u} = (u_1, u_2, u_3)$  and  $\vec{v} = (v_1, v_2, v_3)$  are vectors in  $\mathbb{R}^3$  and that  $\theta \in [0, \pi]$  represents the angle between them. We define the scalar product  $\vec{u} \cdot \vec{v}$  of  $\vec{u}$  and  $\vec{v}$  by

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$$

Alternatively, we can write

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

### Theorem 5 (SCALAR PRODUCT)

Suppose that  $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$  and  $c \in \mathbb{R}$ , then

i  $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

ii  $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$

iii  $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$

iv  $\vec{u} \cdot \vec{u} \geq 0$

v  $\vec{u} \cdot \vec{u} = 0$  if and only if  $\vec{u} = 0$

### Remark

- i The *scalar product* is also known as the *dot product* or the *inner product* of  $\vec{u}$  and  $\vec{v}$ .
- ii We say that two non-zero vectors in  $\mathbb{R}^3$  are *orthogonal* if the angle between them is  $\frac{\pi}{2}$ . It follows immediately from the definition of scalar product that any two non-zero vectors  $\vec{u}, \vec{v} \in \mathbb{R}^3$  are orthogonal if and only if  $\vec{u} \cdot \vec{v} = 0$ .
- iii Using the definition of scalar product we can calculate the angle between  $\vec{u}$  and  $\vec{v}$  since

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

**Example** Suppose  $\vec{u} = (2, 4, 6)$  and  $\vec{v} = (1, -2, 3)$ . Then

$$\begin{aligned} \vec{u} \cdot \vec{v} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= 2 \cdot 1 + 4 \cdot (-2) + 6 \cdot 3 \\ &= 12 \end{aligned}$$

**Example** Suppose  $\vec{u} = (2, 0, 0)$  and  $\vec{v} = (1, 1, \sqrt{2})$ . Then we have  $\vec{u} \cdot \vec{v} = 2$ . Note now that  $\|\vec{u}\| = 2$  and  $\|\vec{v}\| = 2$ . It follows that

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{2}{4} = \frac{1}{2}$$

Hence

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$

**Example** Suppose  $\vec{u} = (-4, -1, 1)$  and  $\vec{v} = (1, -2, 5)$ . Then we have  $\vec{u} \cdot \vec{v} = 3$ . Note now that  $\|\vec{u}\| = \sqrt{18}$  and  $\|\vec{v}\| = \sqrt{30}$ . It follows that

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{3}{\sqrt{18} \cdot \sqrt{30}} = \frac{3}{23 \cdot 24}$$

Hence

$$\theta = \cos^{-1}\left(\frac{3}{23 \cdot 24}\right) = 82 \cdot 58^\circ$$

**Example** Suppose  $\vec{u} = (2, 3, 5)$  and  $\vec{v} = (1, 1, -1)$ . Then we have  $\vec{u} \cdot \vec{v} = 0$ . It follows that  $\vec{u}$  and  $\vec{v}$  are orthogonal.

**Exercise** Let  $\vec{u} = (2, 4, -3)$  and  $\vec{v} = (8, -1, -1)$ . Determine the *scalar product*  $\vec{u} \cdot \vec{v}$  and hence determine the angle between  $\vec{u}$  and  $\vec{v}$ .

**Exercise** Let  $\vec{u} = (1, -2, -5)$  and  $\vec{v} = (0, -1, -1)$ . Determine the *scalar product*  $\vec{u} \cdot \vec{v}$  and hence determine the angle between  $\vec{u}$  and  $\vec{v}$ .

**Exercise** Let  $\vec{u} = (5, 6, -9)$  and  $\vec{v} = (1, -1, -1)$ . Determine the *scalar product*  $\vec{u} \cdot \vec{v}$  and hence determine the angle between  $\vec{u}$  and  $\vec{v}$ .

## 2.5 Components and Projections

**Definition** Let  $\vec{u}$  and  $\vec{v}$  be two non-zero vectors and  $\theta$  the angle between them. The *scalar component* of  $\vec{u}$  along  $\vec{v}$  is the number

$$\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$$

\* Since  $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$ , the scalar component may also be written as  $\|\vec{u}\| \cos \theta$ .

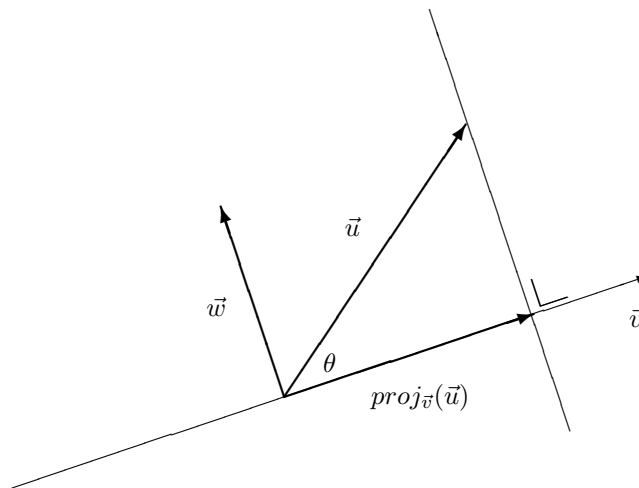
The *vector projection* of  $\vec{u}$  along  $\vec{v}$ , denoted by  $proj_{\vec{v}}(\vec{u})$ , is the vector defined by

$$proj_{\vec{v}}(\vec{u}) = \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v}$$

that is, the scalar multiple of the direction of  $\vec{v}$  by the scalar component of  $\vec{u}$  along  $\vec{v}$ .

### Remark

- i In some applications it can be useful to decompose or resolve a vector  $\vec{u}$  into two vectors – one parallel to non-zero vector  $\vec{v}$  and the other perpendicular to  $\vec{v}$ .



To resolve a given vector  $\vec{u}$  into two vectors – one parallel to a given non-zero vector  $\vec{v}$  and the other perpendicular to the vector  $\vec{v}$  we first calculate  $proj_{\vec{v}}(\vec{u})$ , the *vector projection* of  $\vec{u}$  along  $\vec{v}$  and secondly a perpendicular vector to  $\vec{v}$ , which we will label  $\vec{w}$  and is given as

$$\vec{w} = \vec{u} - proj_{\vec{v}}(\vec{u})$$

The resultant of  $proj_{\vec{v}}(\vec{u})$  and  $\vec{w}$  will yield  $\vec{u}$ . Furthermore, their *scalar product* is zero.

- ii As the name suggests, the *scalar component* is a scalar and the *vector projection* is a vector.  
 iii The  $proj_{\vec{v}}(\vec{u})$  has the same direction as  $\vec{v}$  if  $\theta$  is acute, and the opposite direction if  $\theta$  is obtuse.  
 iv The length or magnitude of  $proj_{\vec{v}}(\vec{u})$  is

$$\left| \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \right|$$

that is, the absolute value of the scalar component of  $\vec{u}$  along  $\vec{v}$ .



**Example** Let  $\vec{u} = (2, -1, 3)$  and  $\vec{v} = (1, -3, -1)$ .

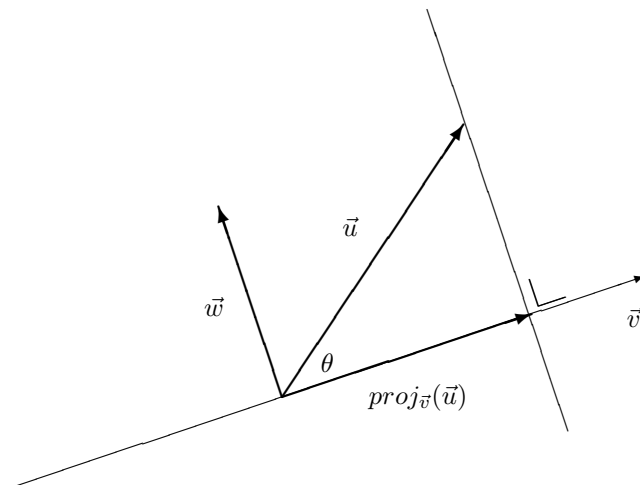
To determine the *vector projection* of  $\vec{u}$  along  $\vec{v}$  we have

$$\text{proj}_{\vec{v}}(\vec{u}) = \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v}$$

Hence

$$\text{proj}_{\vec{v}}(\vec{u}) = \left( \frac{2 + 3 - 3}{\sqrt{11}} \right) \cdot \frac{(1, -3, -1)}{\sqrt{11}} = \frac{2}{11}(1, -3, -1)$$

**Note** Consider the example above with the diagram depicting a vector projection. From the vector  $\vec{u}$  we have determined  $\text{proj}_{\vec{v}}(\vec{u})$ , the vector projection of  $\vec{u}$  along  $\vec{v}$ .



We can determine the vector  $\vec{w}$  as follows

$$\vec{w} = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = (2, -1, 3) - \frac{2}{11}(1, -3, -1) = \frac{5}{11}(4, -1, 7)$$

The resolvent of  $\text{proj}_{\vec{v}}(\vec{u})$  and  $\vec{w}$  should yield  $\vec{u}$ .

$$\text{proj}_{\vec{v}}(\vec{u}) + \vec{w} = \frac{2}{11}(1, -3, -1) + \frac{5}{11}(4, -1, 7) = \left( \frac{22}{11}, \frac{-11}{11}, \frac{33}{11} \right) = (2, -1, 3)$$

Finally, their *scalar product* is zero.

$$\text{proj}_{\vec{v}}(\vec{u}) \cdot \vec{w} = \frac{2}{11}(1, -3, -1) \cdot \frac{5}{11}(4, -1, 7) = \frac{40}{11} + \frac{30}{11} - \frac{70}{11} = 0$$

as expected.

**Exercise** Let  $\vec{u} = (2, 0, 1)$  and  $\vec{v} = (3, 1, -2)$ . Determine the *vector projection* of  $\vec{u}$  along  $\vec{v}$

**Exercise** Let  $\vec{u} = (-1, 2, 4)$  and  $\vec{v} = (0, 1, -6)$ . Determine the *vector projection* of  $\vec{u}$  along  $\vec{v}$

**Exercise** Let  $\vec{u} = (2, 2, 7)$  and  $\vec{v} = (3, 6, -5)$ . Determine the *vector projection* of  $\vec{u}$  along  $\vec{v}$

**Exercise** Let  $\vec{u} = (2, 1, 2)$  and  $\vec{v} = (6, -1, 0)$ . Resolve the vector  $\vec{u}$  into vectors parallel and perpendicular to the vector  $\vec{v}$ .

**Exercise** Let  $\vec{u} = (3, 4, 5)$  and  $\vec{v} = (1, 1, -2)$ . Resolve the vector  $\vec{u}$  into vectors parallel and perpendicular to the vector  $\vec{v}$ .

## 2.6 The Vector Product

We now discuss a product of vectors unique to  $\mathbb{R}^3$ . The idea of vector products has a wide applications in geometry, physics and engineering, and is motivated by the wish to find a vector that is perpendicular to two given vectors.

**Definition** Suppose that  $\vec{u} = (u_1, u_2, u_3)$  and  $\vec{v} = (v_1, v_2, v_3)$  are vectors in  $\mathbb{R}^3$  and that  $\theta \in [0, \pi]$  represents the angle between them. Let  $\vec{n}$  be a unit vector perpendicular to both  $\vec{u}$  and  $\vec{v}$ . Then the *vector product* (or cross product) of  $\vec{u}$  and  $\vec{v}$  is the vector denoted by  $\vec{u} \times \vec{v}$  and defined by

$$\vec{u} \times \vec{v} = \|\vec{u}\| \|\vec{v}\| \sin \theta \vec{n}$$

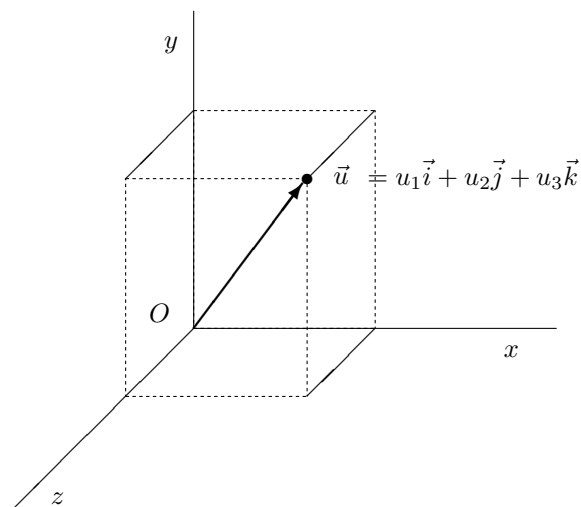
Alternatively

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

**Remark** The vector product  $\vec{u} \times \vec{v}$  yields a vector in  $\mathbb{R}^3$ . In order to develop this component representation of  $\vec{u} \times \vec{v}$  we will switch momentarily from the component representation of a vector  $\vec{u} = (u_1, u_2, u_3)$  to its equivalent cartesian form

$$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$$

where the three unit vectors  $\vec{i} = (1, 0, 0)$ ,  $\vec{j} = (0, 1, 0)$  and  $\vec{k} = (0, 0, 1)$ , the unit vectors along the x,y and z-axes respectively.



Suppose

$$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$$

$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

$$\begin{aligned}\vec{u} \times \vec{v} &= (u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) \times (v_1\vec{i} + v_2\vec{j} + v_3\vec{k}) \\ &= u_1v_1\vec{i} \times \vec{i} + u_1v_2\vec{i} \times \vec{j} + u_1v_3\vec{i} \times \vec{k} \\ &\quad + u_2v_1\vec{j} \times \vec{i} + u_2v_2\vec{j} \times \vec{j} + u_2v_3\vec{j} \times \vec{k} \\ &\quad + u_3v_1\vec{k} \times \vec{i} + u_3v_2\vec{k} \times \vec{j} + u_3v_3\vec{k} \times \vec{k}\end{aligned}$$

Using each of the following facts

$$\vec{u} \times \vec{v} = \|\vec{u}\|\|\vec{v}\|\sin\theta\vec{n}$$

$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$

we can establish each of the following

$$\begin{array}{lll}\vec{i} \times \vec{i} = \vec{0} & \vec{i} \times \vec{j} = \vec{k} & \vec{j} \times \vec{i} = -\vec{k} \\ \vec{j} \times \vec{j} = \vec{0} & \vec{j} \times \vec{k} = \vec{i} & \vec{i} \times \vec{k} = -\vec{j} \\ \vec{k} \times \vec{k} = \vec{0} & \vec{k} \times \vec{i} = \vec{j} & \vec{k} \times \vec{j} = -\vec{i}\end{array}$$

Hence, we have

$$\begin{aligned}\vec{u} \times \vec{v} &= u_1v_1(0) + u_1v_2\vec{k} + u_1v_3(-\vec{j}) \\ &\quad + u_2v_1(-\vec{k}) + u_2v_2(0) + u_2v_3\vec{i} \\ &\quad + u_3v_1\vec{j} + u_3v_2(-\vec{i}) + u_3v_3(0) \\ &= (u_2v_3 - u_3v_2)\vec{i} + (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}\end{aligned}$$

Finally, returning to our component representation we have

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

**Remark** A convenient way of determining the vector product  $\vec{u} \times \vec{v}$  is as follows

$$\vec{u} \times \vec{v} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

Using the cofactor expansion by row 1, we have

$$\begin{aligned} \vec{u} \times \vec{v} &= \det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} \vec{i} - \det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} \vec{j} + \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \vec{k} \\ &= \left( \det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix}, -\det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix}, \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \right) \\ &= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \end{aligned}$$

We will first show that the vector product  $\vec{u} \times \vec{v}$  is orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

### Theorem 6

Suppose that  $\vec{u} = (u_1, u_2, u_3)$  and  $\vec{v} = (v_1, v_2, v_3)$  are vectors in  $\mathbb{R}^3$ . Then

$$i. \vec{u} \cdot (\vec{u} \times \vec{v}) = 0$$

$$ii. \vec{v} \cdot (\vec{u} \times \vec{v}) = 0$$

**Example** Suppose that  $\vec{u} = (1, -1, 2)$  and  $\vec{v} = (3, 0, 2)$ . Then

$$\begin{aligned} \vec{u} \times \vec{v} &= \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 3 & 0 & 2 \end{pmatrix} \\ &= \left( \det \begin{pmatrix} -1 & 2 \\ 0 & 2 \end{pmatrix}, -\det \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \det \begin{pmatrix} 1 & -1 \\ 3 & 0 \end{pmatrix} \right) \\ &= (-2 + 0, -(2 - 6), 0 + 3) \\ &= (-2, 4, 3) \end{aligned}$$

Note that  $(1, -1, 2) \cdot (-2, 4, 3) = 0$  and  $(3, 0, 2) \cdot (-2, 4, 3) = 0$ .

**Exercise** For the vectors  $\vec{u} = (1, 2, 3)$  and  $\vec{v} = (3, 2, 1)$  in  $\mathbb{R}^3$ , evaluate

i  $\vec{u} \times \vec{v}$

ii  $\vec{v} \times \vec{u}$

What comment can you make about your answer.

**Theorem 7 (VECTOR PRODUCT)**

Suppose that  $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$  and  $c \in \mathbb{R}$ . Then

i  $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ ;

ii  $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$ ;

iii  $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$ ;

iv  $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$ ;

v  $\vec{u} \times \mathbf{0} = \mathbf{0}$ ;

vi  $\vec{u} \times \vec{u} = \mathbf{0}$ .

Now to consider an application of vector product – to evaluate the area of a parallelogram. To do this we first establish the following result.

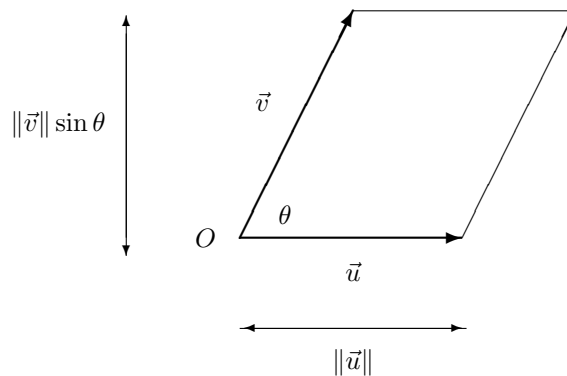
**Theorem 8**

Suppose that  $\vec{u} = (u_1, u_2, u_3)$  and  $\vec{v} = (v_1, v_2, v_3)$  are non-zero vectors in  $\mathbb{R}^3$ , and that  $\theta \in [0, \pi]$  represents the angle between them. Then

i  $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$

ii  $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$

Now consider a parallelogram below.



The base of the parallelogram is given by  $\|\vec{u}\|$ , and hence the height of the parallelogram is given as  $\|\vec{v}\| \sin \theta$ . Therefore, from theorem 6 we can say that the area of the parallelogram is given by  $\|\vec{u} \times \vec{v}\|$ .

### Theorem 9

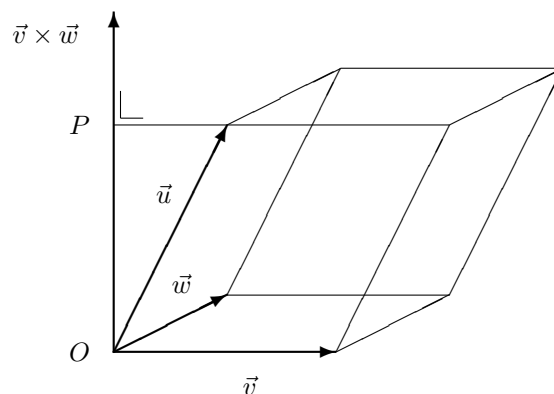
Suppose that  $\vec{u}, \vec{v} \in \mathbb{R}^3$ . Then the parallelogram with  $\vec{u}$  and  $\vec{v}$  as two of its sides has area  $\|\vec{u} \times \vec{v}\|$ .

**Exercise** Let  $\vec{u} = (1, 1, -4)$  and  $\vec{v} = (4, 1, 7)$  in  $\mathbb{R}^3$ .

Determine the area of the *parallelogram* that is defined by  $\vec{u}$  and  $\vec{v}$ .

## 2.7 Scalar Triple Product

Now suppose that  $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$  that do **not** lie all on the same plane. What is formed is a parallelepiped, i.e. a solid body in which each face is a parallelogram, with  $\vec{u}, \vec{v}$  and  $\vec{w}$  as three of its edges.



The base of the parallelepiped has area  $\|\vec{v} \times \vec{w}\|$

If the vector  $OP$  is perpendicular to the base of the parallelepiped, then  $OP$  is in the direction of  $\vec{v} \times \vec{w}$ . Now the height of the parallelepiped is equal to the norm of the orthogonal projection of  $\vec{u}$  on  $\vec{v} \times \vec{w}$ . In other words, the parallelepiped has height

$$\text{proj}_{\vec{v} \times \vec{w}}(\vec{u}) = \left( \frac{\vec{u} \cdot (\vec{v} \times \vec{w})}{\|\vec{v} \times \vec{w}\|} \right) \frac{\vec{v} \times \vec{w}}{\|\vec{v} \times \vec{w}\|}$$

Hence, we have

$$\|\text{proj}_{\vec{v} \times \vec{w}}(\vec{u})\| = \frac{\vec{u} \cdot (\vec{v} \times \vec{w})}{\|\vec{v} \times \vec{w}\|}$$

Therefore the volume of the parallelepiped is given by

$$V = \vec{u} \cdot (\vec{v} \times \vec{w})$$

**Theorem 10**

Suppose that  $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ . Then the parallelepiped with  $\vec{u}, \vec{v}$  and  $\vec{w}$  as three of its edges has volume  $\vec{u} \cdot (\vec{v} \times \vec{w})$ .

**Definition** Suppose that  $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ . Then  $\vec{u} \cdot (\vec{v} \times \vec{w})$  is called the scalar triple product of  $\vec{u}, \vec{v}$  and  $\vec{w}$ .

**Remark** It follows from theorem 9 that three vectors in  $\mathbb{R}^3$  are *coplanar* if and only if their scalar triple product is zero.

**Example** Suppose that  $\vec{u} = (1, 0, 1)$ ,  $\vec{v} = (2, 1, 3)$  and  $\vec{w} = (0, 1, 1)$ . Then

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \det \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix} = 0$$

Hence  $\vec{u}, \vec{v}$  and  $\vec{w}$  are coplanar.

**Example** The volume of the parallelepiped with  $\vec{u} = (1, 0, 1)$ ,  $\vec{v} = (2, 1, 4)$  and  $\vec{w} = (0, 1, 1)$  as three of its edges are given by

$$V = \vec{u} \cdot (\vec{v} \times \vec{w}) = \det \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix} = -1$$

We take the absolute value 1.

**Exercise** Let  $\vec{u} = (2, 1, 3)$ ,  $\vec{v} = (4, -1, 0)$  and  $\vec{w} = (2, 0, 1)$ .

Show that  $\vec{u}, \vec{v}$  and  $\vec{w}$  are *coplanar*.

**Exercise** Let  $\vec{u} = (5\lambda, 2\lambda, 3)$ ,  $\vec{v} = (1, 1, 0)$  and  $\vec{w} = (0, 2, -1)$ .

For which values of  $\lambda$  are these vectors *coplanar*?

## 2.8 Some Exercises

**Exercise** Consider the following vectors  $\vec{u} = (1, 2, 3)$  and  $\vec{v} = (3, 2, 1)$  in  $\mathbb{R}^3$ .

- i Evaluate  $\vec{u} - 4\vec{v}$
- ii Evaluate  $\vec{u} \cdot \vec{v}$
- iii Determine  $\|7\vec{u} - 2\vec{v}\|$
- iv Determine  $\vec{u} \times \vec{v}$ .
- v Determine the *vector projection* of  $\vec{u}$  along  $\vec{v}$ , i.e.,  $proj_{\vec{v}}(\vec{u})$

**Exercise** Consider the following vectors  $\vec{u} = (1, 0, 1)$ ,  $\vec{v} = (2, 1, 3)$  and  $\vec{w} = (0, 1, 1)$  in  $\mathbb{R}^3$ .

- i Evaluate  $\vec{u} - 4\vec{v} + 2\vec{w}$
- ii Determine  $\|7\vec{u} - 2\vec{v}\|$
- iii Evaluate  $\vec{u} \cdot \vec{v}$
- iv Determine the angle  $\theta$  between  $\vec{u}$  and  $\vec{v}$ .
- v Determine the *vector projection* of  $\vec{u}$  along  $\vec{v}$ , i.e.,  $proj_{\vec{v}}(\vec{u})$
- vi Calculate the components  $a, b$  and  $c$  of some non-zero vector that is orthogonal to  $\vec{u}$  and  $\vec{v}$ .
- vii Determine the area of the *parallelogram* that is defined by  $\vec{u}$  and  $\vec{v}$ .
- viii Determine  $\vec{u} \cdot (\vec{v} \times \vec{w})$ . What comment can you make about the vectors  $\vec{u}, \vec{v}$  and  $\vec{w}$ ?

**Exercise** Find the interior angles  $\alpha, \beta, \gamma$  of a triangle ABC whose vertices are the points

$$A(-1, 0, 2) \quad , \quad B(2, 1, -1) \quad , \quad C(1, -2, 2)$$



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