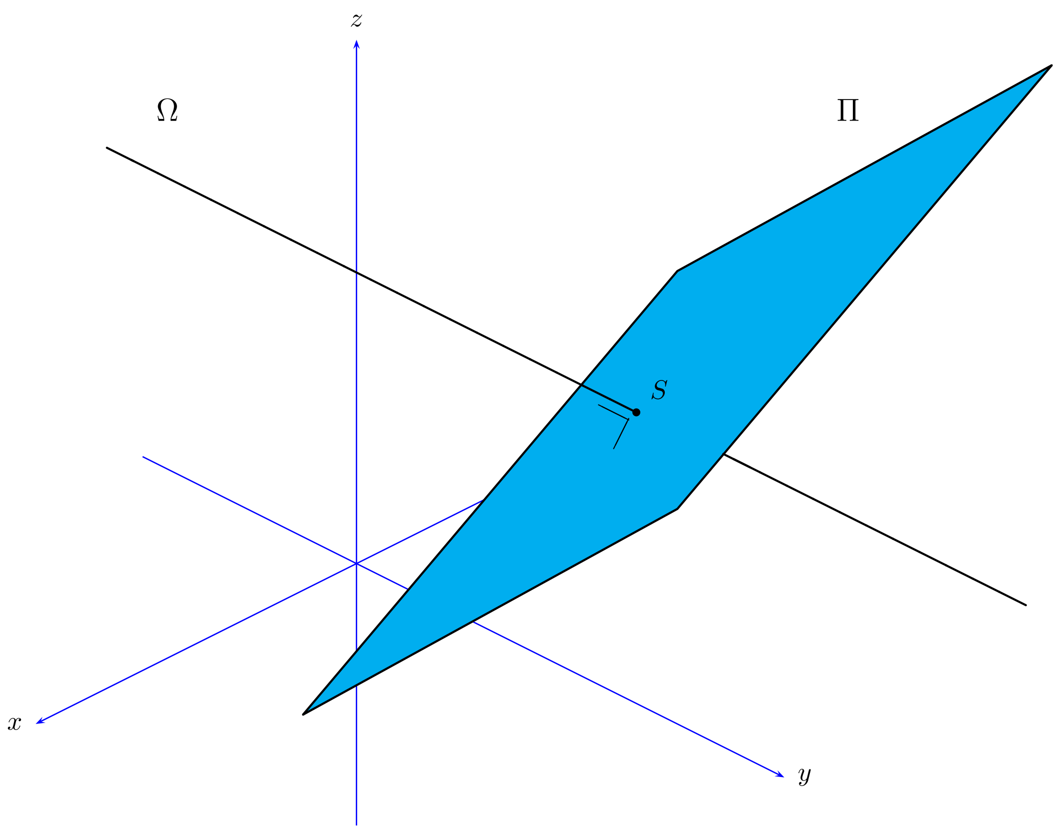
INSTITIÚID TEICNEOLAÍOCHTA CHEATHARLACH

INSTITUTE OF TECHNOLOGY CARLOW

GEOMETRY IN THREE DIMENSIONS



Contents

1 Geometry in $\mathbb{R}^3$

We have studied some relationships between algebra and geometry. We now use this algebra to better understand some problems in geometry. Vectors can be used to derive equations for lines, curves and surfaces. When talking about the equation of a line, curve or surface one talks about defining in terms of an *implicit equation* or by a *parametric equation*.

1.1 Lines

Definition 1 *Let L be a line in $\mathbb{R}^3$.*

1. *Find a point $P_0(x_0, y_0, z_0)$ which is on L.*
2. *Find a vector $\vec{v} = (v_1, v_2, v_3)$ which is parallel to L*
3. *Then $t\vec{v}$ is also parallel to L and the line is the set of all points $P(x, y, z)$ for which the vector*

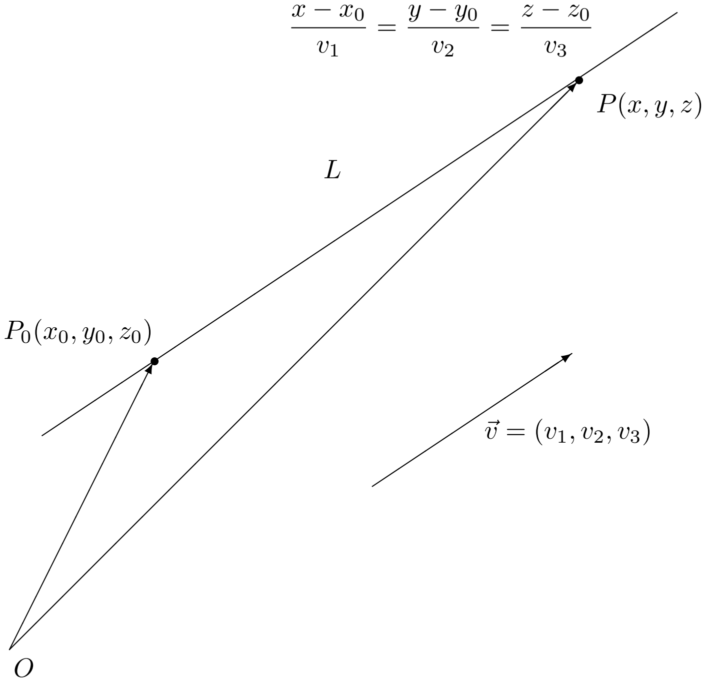
$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0) = t\vec{v} = (tv_1, tv_2, tv_3)$$

for some $t \in \mathbb{R}$. Expanding gives

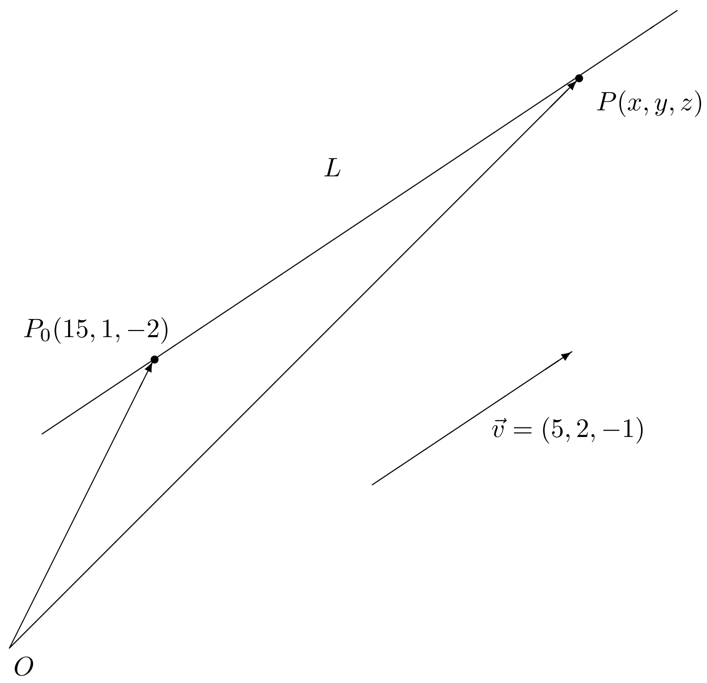
$$\begin{aligned} x &= x_0 + tv_1 \\ y &= y_0 + tv_2 \\ z &= z_0 + tv_3 \end{aligned}$$

These are the *parametric equations* of the line L through the point $P_0 = (x_0, y_0, z_0)$ in the direction of the vector $\vec{v} = (v_1, v_2, v_3)$. Here t is called the parameter. We may eliminate t from the equations to get the *Cartesian equation* of L.

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$



Example Find the line through $P_0(15,1,-2)$ parallel to $\vec{v}=(5,2,-1)$



Solution:

Let $P_0(15,1,-2)$ which is on L.

Let $\vec{v}=(5,2,-1)$ which is parallel to L.

Then $t\vec{v}$ is then also parallel to L and the line is the set of all points $P(x,y,z)$ for which the vector

$$\overrightarrow{P_0P}=(x-15,y-1,z+2)=t\vec{v}=(5t,2t,-t)$$

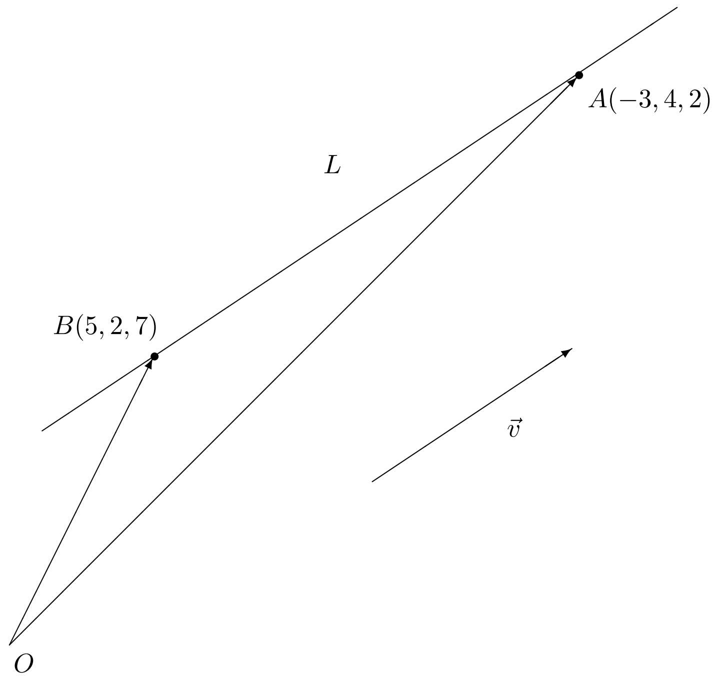
for some $t\in\mathbb{R}$. Expanding gives

$$\begin{aligned} x &= 15+5t \\ y &= 1+2t \\ z &= -2-t \end{aligned}$$

Eliminating t from the *parametric equations* yield the *Cartesian equation* for L

$$\frac{x-15}{5}=\frac{y-1}{2}=\frac{z+2}{-1}$$

Example To find the line L through $A(-3,4,2)$ and $B(5,2,7)$



Solution:

Let L be a line in $\mathbb{R}^3$ passing through the points $A(-3,4,2)$ and $B(5,2,7)$.

Let $P_0(-3,4,2)$ which is on L.

Let $\vec{v} = \overrightarrow{AB}$ which is parallel to L. Now

$$\begin{aligned} \vec{v} &= \overrightarrow{AB} \\ &= \vec{B} - \vec{A} \\ &= (5,2,7) - (-3,4,2) \\ &= (8,-2,5) \end{aligned}$$

Then $t\vec{v}$ is then also parallel to L and the line is the set of all points $P(x,y,z)$ for which the vector

$$\overrightarrow{P_0P} = (x+3, y-4, z-2) = t\vec{v} = (8t, -2t, 5t)$$

for some $t \in \mathbb{R}$. Expanding gives

$$\begin{aligned} x &= -3+8t \\ y &= 4-2t \\ z &= 2+5t \end{aligned}$$

Eliminating t from the *parametric equations* yield the *Cartesian equation* for L

$$\frac{x+3}{8} = \frac{y-4}{-2} = \frac{z-2}{5}$$

Remark In this previous example we found the line L through $A(-3,4,2)$ and $B(5,2,7)$ has parametric equations

$$\begin{aligned} x &= -3+8t \\ y &= 4-2t \\ z &= 2+5t \end{aligned}$$

for some $t \in \mathbb{R}$. However, the same line, using the point B instead of A yield parametric equations

$$\begin{aligned} x &= 5+8u \\ y &= 2-2u \\ z &= 7+5u \end{aligned}$$

for some $u \in \mathbb{R}$.

Both sets of equations describe the same line L since we can set $t = u + 1$ and $u = t - 1$. In general, different choices of parameter will yield different equations for L, and it may be misleading to talk of the equations of L without specifying the point and the vector which determine the parameter.

Exercise Consider the points $P(2,3,1)$ and $Q(4,2,5)$. Find the equation of the line through P and Q.

1.2 Planes

Definition 2 *A plane is defined as a set of points whose coordinates* (x, y, z) *satisfy a condition of the form*

$$f(x, y, z) = 0$$

where f *is a function of* x, y *and* z*. This is the implicit representation of the plane.*

A plane in $\mathbb{R}^3$ can be defined by specifying its inclination and one of its points (much in the same way as a line can be defined by specifying its slope and one of its points). An easy way to describe the inclination is to specify a non-zero vector that is perpendicular to the plane – such a vector is called a *normal.* The implicit equation of the plane will be of the form

$$ax + by + cz + d = 0$$

The general method for finding the implicit equation of a plane is:

1. Find the co-ordinates of a point $P_0(x_0, y_0, z_0)$ which is on the plane.
2. Let $\vec{n} = (a, b, c)$ perpendicular to the plane.
3. Then the plane consists of those points $P(x, y, z)$ for which the vector

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

is orthogonal to $\vec{n}$ i.e.,

$$\overrightarrow{P_0P}.\vec{n} = 0$$

Rewriting this in terms of the components we get

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

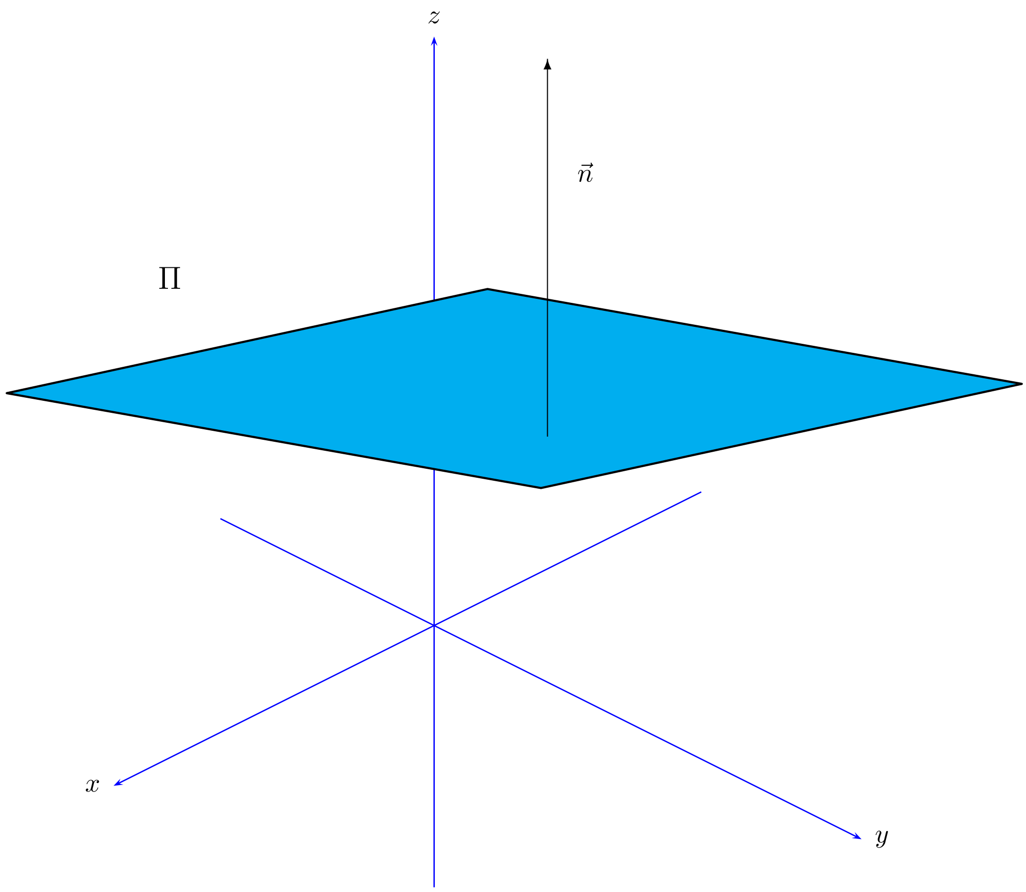
Therefore

$$ax + by + cz + d = 0$$

where $d = -ax_0 - by_0 - cz_0$.

This is called the *point normal* form of the equation of the plane.

Example Find the equation of the plane Π through $P_0(1,3,2)$ with normal $\vec{n} = (5,0,-1)$.



Solution:

Let $P_0(1,3,2)$ which is on the plane.

Let $\vec{n} = (5,0,-1)$ perpendicular to the plane.

Then the plane Π consists of those points $P(x,y,z)$ for which the vector

$$\overrightarrow{P_0P} = (x-1, y-3, z-2)$$

is orthogonal to $\vec{n}$ i.e.,

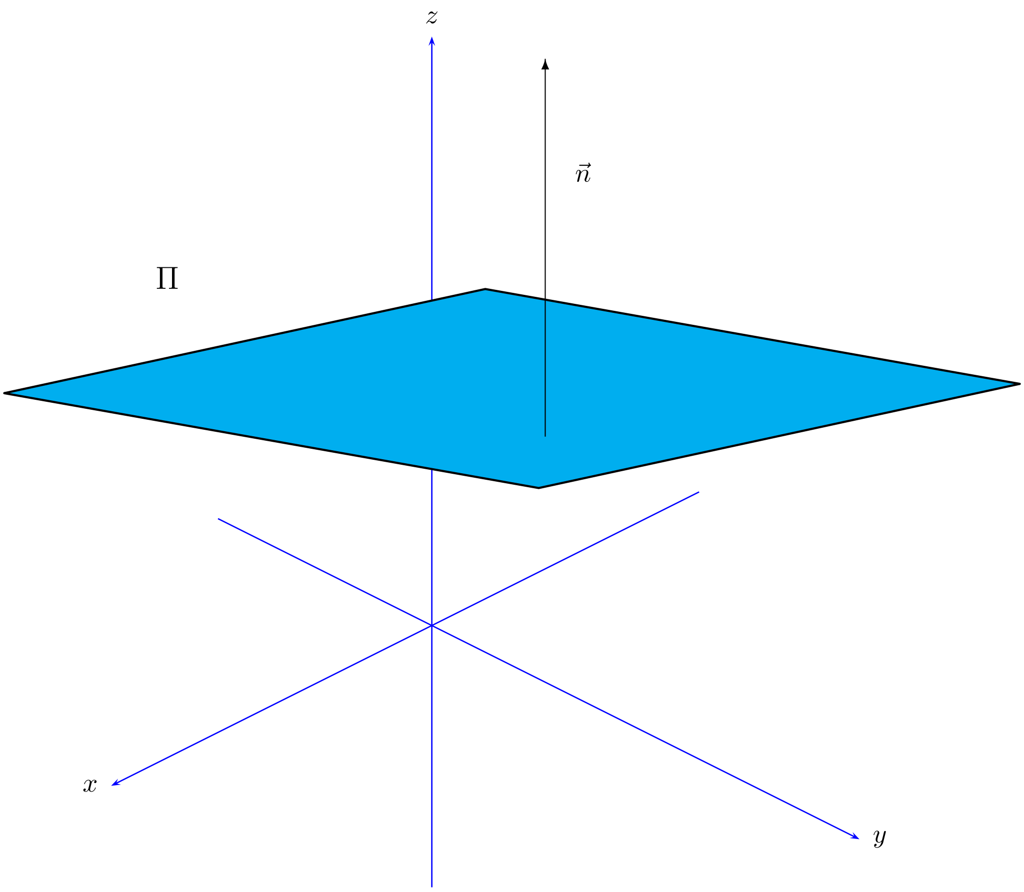
$$\overrightarrow{P_0P}.\vec{n} = 0$$

So that the plane Π has implicit equation

$$5(x-1) + 0(y-3) - 1(z-2) = 0$$

$$\Pi : \quad 5x - z - 3 = 0$$

Example Find the plane Π through $A(1, 0, 2)$, $B(3, -2, 4)$ and $C(6, 2, 1)$.



Solution:

We determine the normal vector $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ which is perpendicular to the plane.

$$\begin{aligned} \overrightarrow{AB} &= (2, -2, 2) \\ \overrightarrow{AC} &= (5, 2, -1) \\ \overrightarrow{AB} \times \overrightarrow{AC} &= (-2, 12, 14) \end{aligned}$$

Then the plane consists of those points $P = (x, y, z)$ for which the vector

$$\overrightarrow{P_0P} = (x - 1, y - 0, z - 2)$$

is orthogonal to $\vec{n}$ i.e.,

$$\overrightarrow{P_0P}.\vec{n} = 0$$

So the plane has implicit equation

$$-2(x-1)+12(y-0)+14(z-2)=0$$

$$\Pi: \quad 2x-12y-14z+26=0$$

Note It is easy to check that this is the correct equation by checking that A, B and C satisfy the equation.

Remark We can make the following important observation. Given the normal vector $\vec{n}=(a,b,c)$ to a plane Π we can write the equation of the plane

$$ax+by+cz+d=0$$

If $P_0 \in \Pi$, then we can solve for the constant d and the finally present the equation of Π. So, for example, in the previous exercise $\vec{n}=(-2,12,14)$. Hence the equation of the plane is

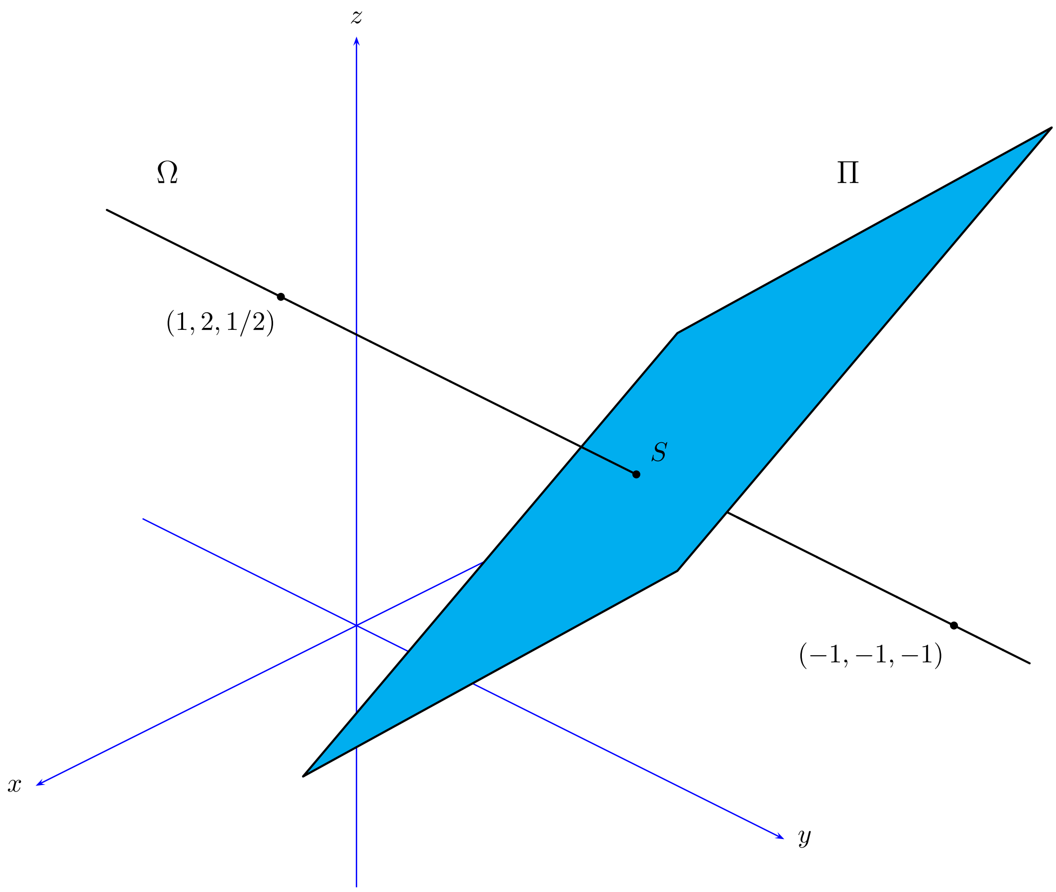
$$-2x+12y+14z+d=0$$

But $A(1,0,2) \in \Pi$, hence $-2(1)+12(0)+14(2)+d=0$. Therefore $d=-26$ and the equation is

$$\Pi: \quad -2x+12y+14z-26=0$$

This procedure presents an alternative solution to determining the equation of a plane in $\mathbb{R}^3$ given a normal vector to the plane and a point on the plane. We refer to this method as a 'Shorthand Method'.

Example The straight line Ω goes through the point $(1, 2, 1/2)$ and $(-1, -1, -1)$. The vector $(-1/4, 1, 2)$ is perpendicular to the plane Π, and Π passes through $(-2, 1, 2)$.



Find the point S at which Ω meets Π.

Solution:

Let $P_0 = (-1, -1, -1)$. The line Ω goes through the points $A(1, 2, \frac{1}{2})$ and $B(-1, -1, -1)$, hence

$$\begin{aligned}\vec{v} &= \overrightarrow{AB} \\ &= \vec{B} - \vec{A} \\ &= (-1, -1, -1) - (1, 2, \frac{1}{2}) \\ &= (-2, -3, -\frac{3}{2})\end{aligned}$$

Now

$$\begin{aligned}\overrightarrow{P_0P} &= \vec{P} - \vec{P_0} \\ &= (x, y, z) - (-1, -1, -1) \\ &= (x + 1, y + 1, z + 1)\end{aligned}$$

Now $\overrightarrow{P_0P} = t\vec{v}$, hence

$$\begin{aligned}(x + 1, y + 1, z + 1) &= t(-2, -3, -\frac{3}{2}) \\ &= (-2t, -3t, -\frac{3}{2}t)\end{aligned}$$

for some $t \in \mathbb{R}$. Expanding gives the parametric equations of the line Ω.

$$\begin{aligned}x &= -1 - 2t \\ y &= -1 - 3t \\ z &= -1 - \frac{3}{2}t\end{aligned}$$

Let $P_0(-2, 1, 2)$ and $\vec{n} = (-\frac{1}{4}, 1, 2)$.

(Shorthand Method) The equation of the plane Π is

$$-\frac{1}{4}x + y + 2z + d = 0$$

But $P_0 \in \Pi$, hence

$$-\frac{1}{4}(-2) + 1 + 2(2) + d = 0$$

Therefore $d = -\frac{11}{2}$. Finally the equation of the plane Π is

$$-\frac{1}{4}x + y + 2z - \frac{11}{2} = 0$$

i.e.,

$$x - 4y - 8z + 22 = 0$$

Now to calculate the point of intersection of the line Ω and the plane Π.

$$\begin{aligned}(-1 - 2t) - 4(-1 - 3t) - 8(-1 - \frac{3}{2}t) + 22 &= 0 \\ -1 - 2t + 4 + 12t + 8 + 12t + 22 &= 0 \\ 22t &= -33 \\ t &= -\frac{3}{2}\end{aligned}$$

Hence, the point of intersection is

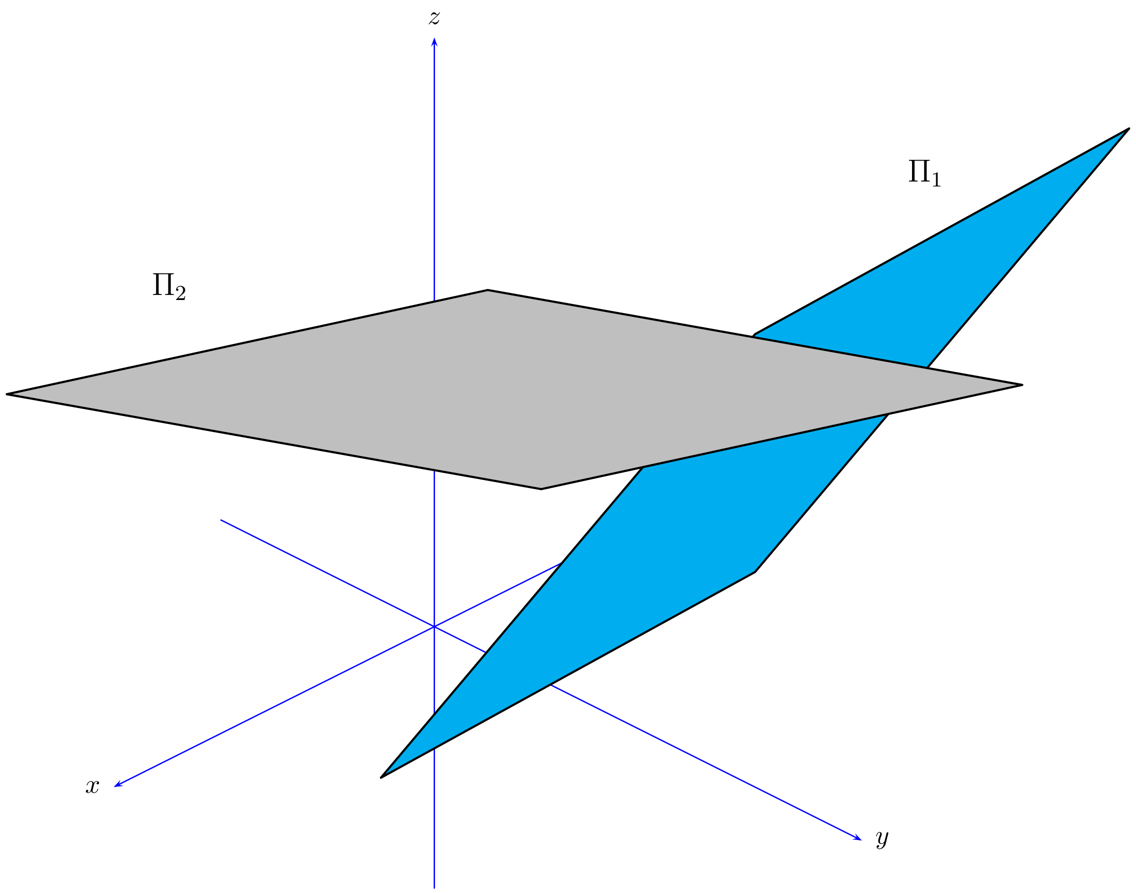
$$\begin{aligned} x &= -1-2(-\frac{3}{2}) = 2 \\ y &= -1-3(-\frac{3}{2}) = \frac{7}{2} \\ z &= -1-\frac{3}{2}(-\frac{3}{2}) = \frac{5}{4} \end{aligned}$$

Therefore, $P = (2, \frac{7}{2}\ , \frac{5}{4})$.

Example

$A(1, 2, 4)$, $B(1, 0, 5)$ and $C(0, -3, 2)$ are points on the plane Π_1.

$A(-1, 1, 2)$, $B(1, 0, 5)$ and $C(0, -2, 6)$ are points on the plane Π_2.



i Determine the equation of Π_1 and Π_2.

ii Find the equation of the line of intersection Ω of the planes Π_1 and Π_2 given that the point $B(1, 0, 5)$ is common to both planes and hence lies along the line of intersection Ω.

iii Determine the angle θ between the planes Π_1 and Π_2.

Solution:

i $A(1,2,4)$, $B(1,0,5)$ and $C(0,-3,2)$ are points on the **plane** Π_1.

Firstly, we require a normal vector to Π_1.

$$\begin{aligned}\overrightarrow{AB} &= \vec{B}-\vec{A}\\ &= (1,0,5)-(1,2,4)\\ &= (0,-2,1)\end{aligned}$$

$$\begin{aligned}\overrightarrow{AC} &= \vec{C}-\vec{A}\\ &= (0,-3,2)-(1,2,4)\\ &= (-1,-5,-2)\end{aligned}$$

$$\vec{n}_1 = \overrightarrow{AB}\times\overrightarrow{AC} = det\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & -2 & 1\\ -1 & -5 & -2\end{vmatrix} = (9,-1,-2)$$

(Shorthand Method) The equation of the plane Π_1 is

$$9x - y - 2z + d = 0$$

But $A(1,2,4) \in \Pi_1$, hence $9(1) - 2 - 2(4) + d = 0$.

Therefore $d = 1$. Finally the equation of the plane Π_1 is

$$9x - y - 2z + 1 = 0$$

$A(-1,1,2)$, $B(1,0,5)$ and $C(0,-2,6)$ are points on the **plane** Π_2.

Firstly, we require a normal vector to Π_2.

$$\begin{aligned}\overrightarrow{AB} &= \vec{B}-\vec{A}\\ &= (1,0,5)-(-1,1,2)\\ &= (2,-1,3)\end{aligned}$$

$$\begin{aligned}\overrightarrow{AC} &= \vec{C}-\vec{A}\\ &= (0,-2,6)-(-1,1,2)\\ &= (1,-3,4)\end{aligned}$$

$$\vec{n_2} = \overrightarrow{AB} \times \overrightarrow{AC} = det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ 1 & -3 & 4 \end{vmatrix} = (5, -5, -5)$$

(Shorthand Method) The equation of the plane Π_2 is

$$5x - 5y - 5z + d = 0$$

But $A(-1, 1, 2) \in \Pi_1$, hence $5(-1) - 5(1) - 5(2) + d = 0$.

Therefore $d = 20$. Finally the equation of the plane Π_2 is

$$5x - 5y - 5z + 20 = 0$$

ii To determine the equation of the line of intersection Ω of the planes Π_1 and Π_2

$$\vec{v} = \vec{n_1} \times \vec{n_2} = det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 9 & -1 & -2 \\ 5 & -5 & -5 \end{vmatrix} = (-5, 35, -40)$$

Also

$$\begin{aligned} \overrightarrow{P_0P} &= \vec{P} - \vec{P_0} \\ &= (x, y, z) - (1, 0, 5) \\ &= (x - 1, y, z - 5) \end{aligned}$$

Now $\overrightarrow{P_0P} = t\vec{v}$, hence

$$\begin{aligned} (x - 1, y, z - 5) &= t(-5, 35, -40) \\ &= (-5t, 35t, -40t) \end{aligned}$$

for some $t \in \mathbb{R}$. Expanding gives the parametric equations of the line of intersection Ω.

$$\begin{aligned} x &= 1 - 5t \\ y &= 35t \\ z &= 5 - 40t \end{aligned}$$

iii To determine the angle θ between the planes Π_1 and Π_2 we need only determine the angle between the normal vectors $\vec{n_1}$ and $\vec{n_2}$ (or the angle between $\overrightarrow{AB}$ on the plane Π_1 and $\overrightarrow{AB}$ on the plane Π_2).

Now $\vec{n_1} = (9, -1, -2)$ and $\vec{n_2} = (5, -5, -5)$

Also $||\vec{n_1}|| = \sqrt{86}$ and $||\vec{n_2}|| = \sqrt{75}$. Hence

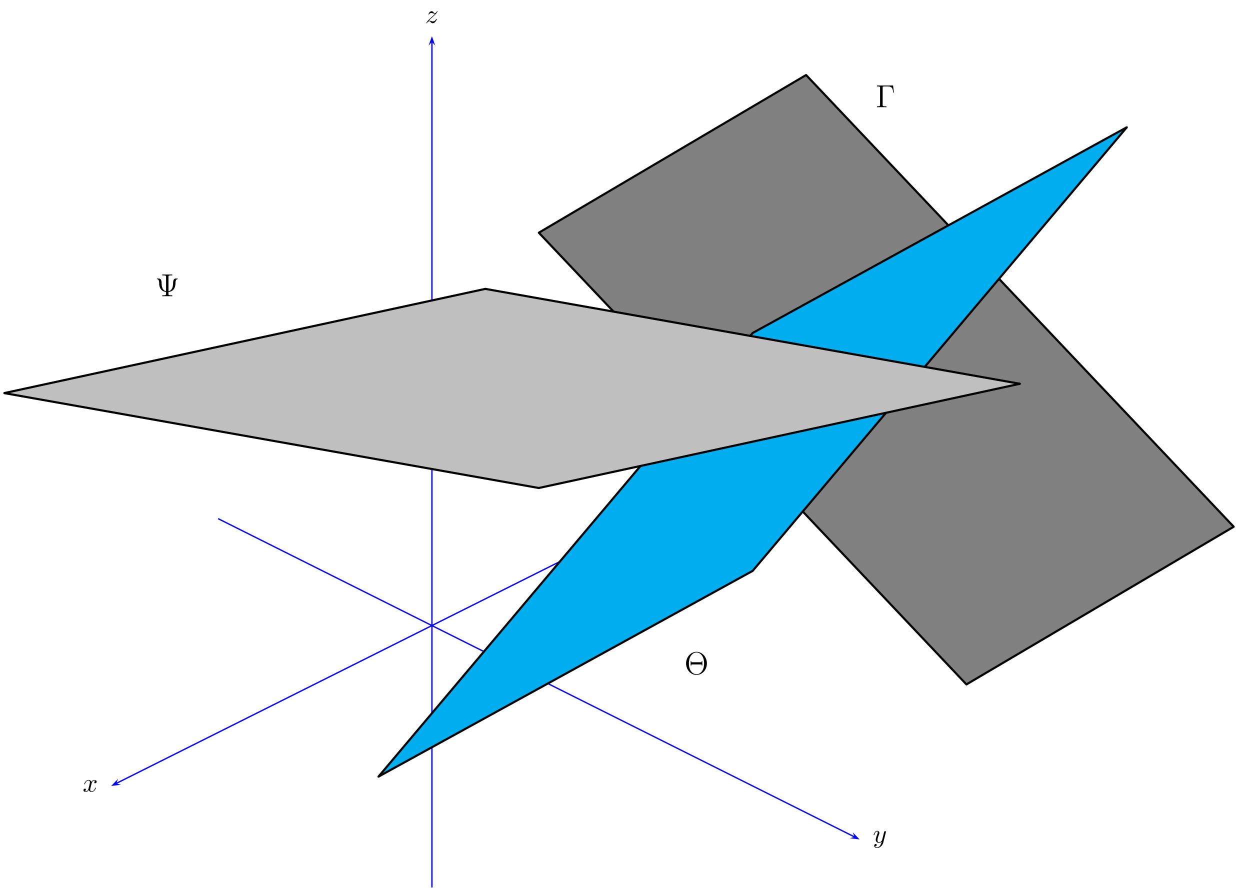
$$\cos\theta = \frac{60}{\sqrt{86}\sqrt{75}}$$

Exercise

i Find the equation of the plane Θ that contains the point $(1, 1, -8)$ that is perpendicular to the line of intersection of the planes

$$\begin{aligned} \Gamma &: 2x + y - 3z = 1 \\ \Psi &: x - y + 3z = 0 \end{aligned}$$

ii By a sequence of *row operations*, determine the point at which the planes Θ, Γ and Ψ intersect.



1.3 Distance from Point to Plane

We turn our attention to the question of finding the distance of a point from a plane.

Theorem 1

The perpendicular distance D of a point $P(x_0, y_0, z_0)$ from the plane $ax + by + cz + d = 0$ is given by

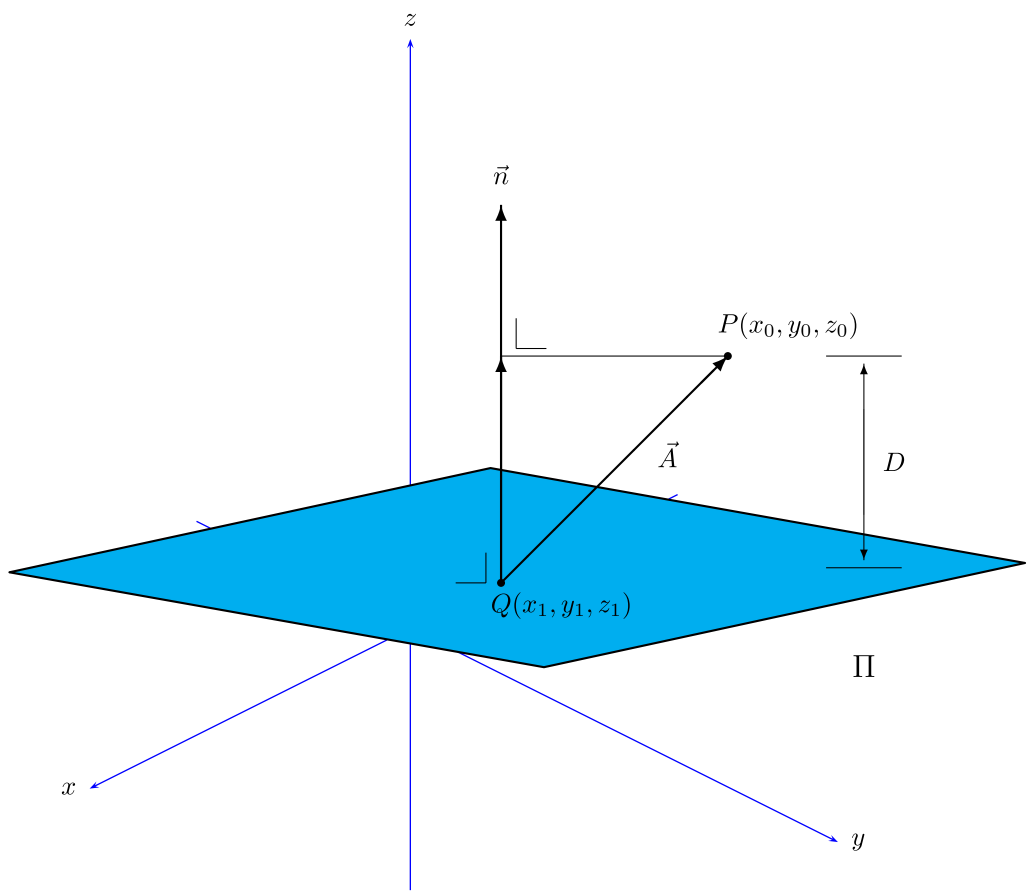
$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Proof Let $\vec{n} = (a, b, c)$ be a normal to the plane.

Let $Q(x_1, y_1, z_1)$ be any point in the plane, so

$$ax_1 + by_1 + cz_1 + d = 0$$

Let $\vec{A} = \overrightarrow{QP} = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$



From the diagram, the required distance D is given by

$$\begin{aligned} D &= |proj_{\vec{n}}(\vec{A})| \\ &= \frac{|\vec{A}.\vec{n}|}{\|\vec{n}\|} \\ &= \frac{|(x_0 - x_1, y_0 - y_1, z_0 - z_1).(a, b, c)|}{\|\vec{n}\|} \\ &= \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\|\vec{n}\|} \\ &= \frac{|ax_0 + by_0 + cz_0 - (ax_1 + by_1 + cz_1)|}{\|\vec{n}\|} \end{aligned}$$

Finally

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Remark When presented with a plane in $\mathbb{R}^3$ we can read off a normal vector to the plane. So, for example, if we have

$$3x + 2y - 5z + 2 = 0$$

we can say a normal vector to the plane is $\vec{n} = (3, 2, -5)$. As remarked earlier, it follows that if we have a normal vector to a plane along with a point on the plane we may easily determine the equation of the plane.

Example Find the equation of the plane which passes through the point $P(2, 3, 4)$ and is parallel to the plane with equation

$$x + 2y - z = 5$$

Solution: Note that the given plane has the vector $(1, 2, -1)$ as a normal, and so also must any plane parallel to the given plane; hence the required plane has an equation of the form $x + 2y - z = d$ for some $d \in \mathbb{R}$. Since $(2, 3, 4)$ lies on the plane, $d = 4$. The required equation of the plane is

$$x + 2y - z = 4$$

** The distance between these planes is the distance from the point $(2, 3, 4)$ to the plane with equation $x + 2y - z = 5$. This is given as

$$D = \frac{|2 + 6 - 4 - 5|}{\sqrt{1 + 4 + 1}} = \frac{1}{\sqrt{6}}$$

1.4 Some Exercises

Exercise Consider the three points $P(2, 3, 1)$, $Q(4, 2, 5)$ and $R(1, 6, -3)$.

i Find the equation of the line through P and Q.

ii Find the equation of the plane perpendicular to the line in part i and passing through the point R.

iii Find the distance between R and the line in part i.

iv Find the area of the parallelogram with the three points as vertices.

v Find the equation of the plane through these points.

vi Find the distance from the origin $(0, 0, 0)$ from the plane in part v.

Exercise Consider the following points in $\mathbb{R}^3$.

$$P(0, -1, -1) \quad , \quad Q(1, -2, 0) \quad , \quad R(2, 0, -2)$$

Find the equation of the plane Π that contain these points.

Exercise Find the equation of the plane that contains the point $(2, 1, -1)$ that is perpendicular to the line of intersection of the planes

$$\begin{aligned} 2x + y - 3z &= 1 \\ x - y + 3z &= 0 \end{aligned}$$

Find the point at which these three planes intersect.

Exercise Find the point of intersection of the following lines in $\mathbb{R}^3$.

$$L_1 : \ x = 3 + 4t \ , \ y = 4 + t \ , \ z = 1$$

$$L_2 : \ x = -1 + 12t \ , \ y = 7 + 6t \ , \ z = 5 + 3t$$

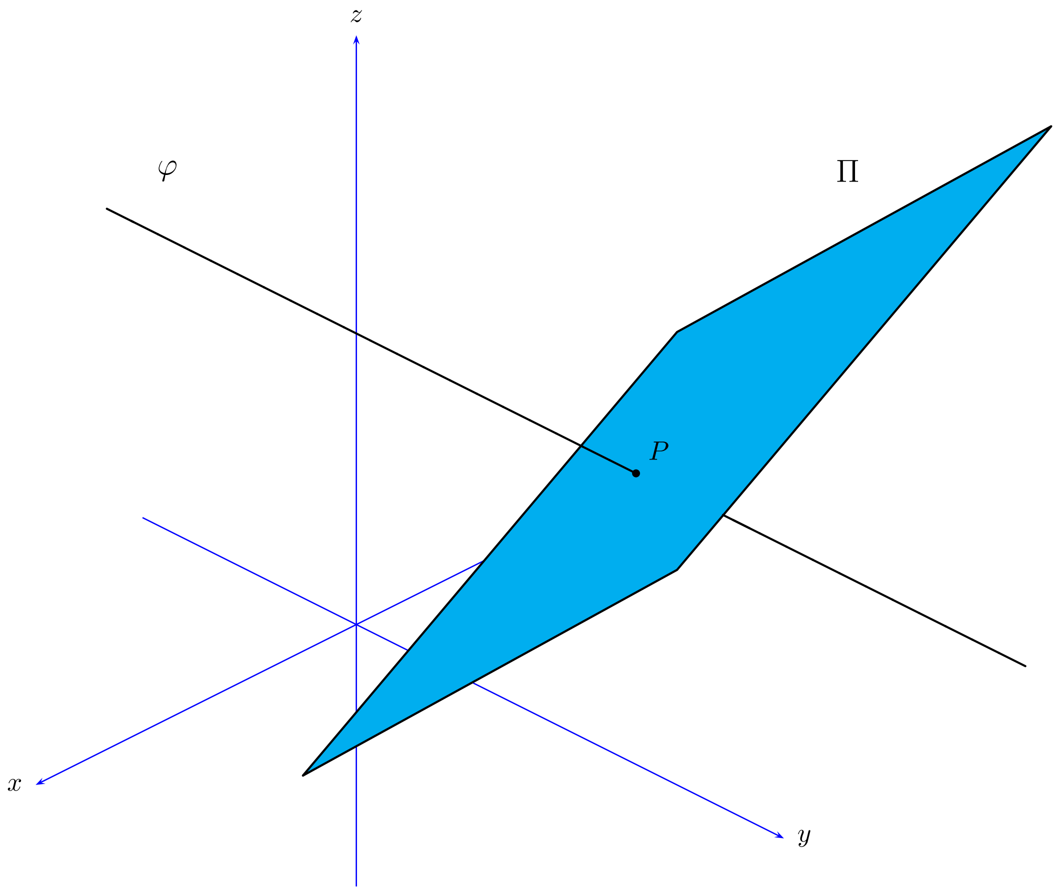
where $t \in \mathbb{R}$.

Exercise

i Find the shortest distance from the point $P(1,1,1)$ to the plane Π that passes through the points $A(0,0,1)$, $B(0,1,-1)$ and $C(-1,0,1)$.

ii Find the equation of the straight line φ that passes through the point $(0,1,0)$ and is parallel to the vector $(-1,1,1)$.

iii Find the point of intersection P of the line φ with the plane Π.



Exercise

i Find a *unit vector* perpendicular to both of the vectors

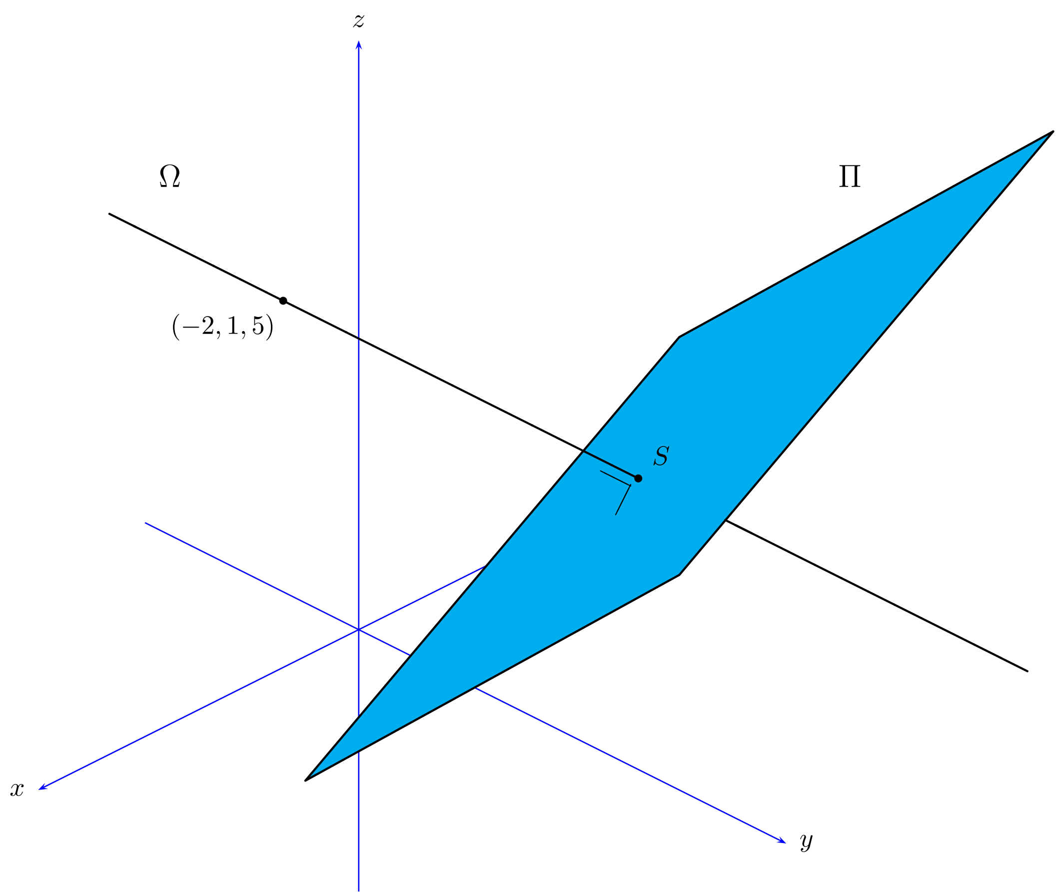
$$P(2,1,-1) \quad , \quad Q(1,-1,2)$$

ii Consider a line Ω through the point $(-2,1,5)$ that is perpendicular to the plane Π with equation $4x-2y+2z+1=0$.

i Find the parametric equations of the line Ω.

ii Find the point S at which Ω meets Π.

iii Determine the perpendicular distance from the point $T(1,2,3)$ to the plane Π.



Exercise

i Find a *point-normal form* of the equation of the plane containing the point $P_0 = (1,1,4)$ with normal vector $\vec{n} = (1,9,8)$.

ii Find the parametric equations of the line of intersection Ω of the planes

$$\begin{aligned} \Gamma &: \quad 2x - 2y + 4z - 6 = 0 \\ \Psi &: \quad -4x - 2y - z - 2 = 0 \end{aligned}$$

Note that $(4,-7,-4) \in \Omega$.