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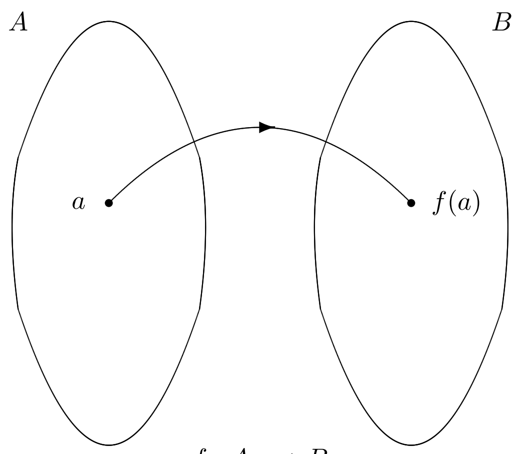
MATRIX TRANSFORMATIONS

1 Matrix Transformations

Definition Let A and B be sets. A *function* $f : A \longrightarrow B$ from A to B assigns to each element a of A a (unique) element $f(a)$ of B. The set A on which the function is defined is referred to as the *domain* of the function $f : A \longrightarrow B$. The set B into which the domain is mapped by f is referred to as the *codomain* of the function f.



$f : A \longrightarrow B$

For many common functions the domain and codomain are the sets of real numbers. We are concerned with functions for which the domain and codomain are *vectors*.

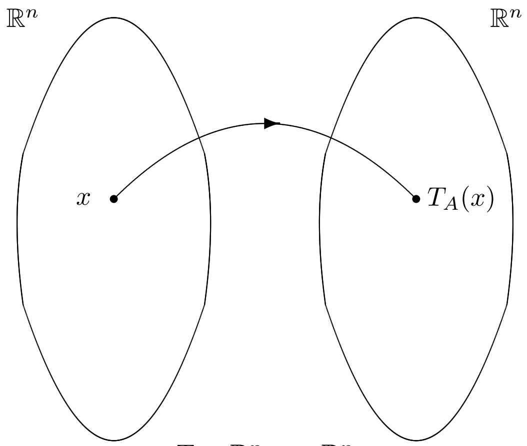
Definition Let A and B be vectors and if f is a *function* with domain A and codomain B, then we say that f is a *transformation* from A to B or that maps A to B, which we denote by writing

$$f : A \longrightarrow B$$

In the special case where $A = B$, the transformation is also called and *operator* on V.

We seek a *transformation* that maps the column vector x in $\mathbb{R}^n$ into the column vector x' in $\mathbb{R}^n$ by multiplying x by the matrix A. We call this a *matrix transformation* and we denote it by

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$



$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

In shorthand, we have

$$T_A(x) = [T]x$$

The matrix transformation T_A is called *multiplication by A*, and the matrix A is called the *standard matrix* for the transformation. The following theorem lists the four basic properties of matrix transformations that follow from properties of matrix multiplication.

Theorem 1

For every matrix A, the matrix transformation $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ has the following properties for all vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ and for every scalar k:

i $T_A(0) = 0$

ii $T_A(k\vec{u}) = kT_A(\vec{u})$

iii $T_A(\vec{u}+\vec{v}) = T_A(\vec{u}) + T_A(\vec{v})$

iv $T_A(\vec{u}-\vec{v}) = T_A(\vec{u}) - T_A(\vec{v})$

The geometric effect of a matrix transformation $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is to map each vector (point) in $\mathbb{R}^n$ into a vector (point) in $\mathbb{R}^m$.

Remark There is a way of finding the *standard matrix* A for a matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ by considering the effect of that transformation on the *standard basis vectors for* $\mathbb{R}^n$.

The *standard basis vectors for* $\mathbb{R}^2$ are $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

The *standard basis vectors for* $\mathbb{R}^3$ are $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

To explain the idea, in general, suppose that the standard matrix A is unknown and that

$$e_1 \ , \ e_2 \ , \ldots\ldots\ldots\ldots\ldots\ldots, \ e_n$$

are the standard basis vectors for $\mathbb{R}^n$. Suppose also that the images of these vectors under the transformation T_A are

$$T_A(e_1) = Ae_1 \quad , \quad T_A(e_2) = Ae_2 \quad , \ldots\ldots\ldots\ldots\ldots\ldots, \quad T_A(e_n) = Ae_n$$

We can write that

$$A = \Big[T_A(e_1) \mid T_A(e_2) \mid \ldots\ldots\ldots\ldots\ldots\ldots \mid T_A(e_n) \Big]$$

In summary, we have the following procedure for finding the standard matrix for a matrix transformation:

Step 1: Find the images of the standard basis vectors $e_1, e_2, \ldots\ldots\ldots\ldots, e_n$ for $\mathbb{R}^n$. in column form.

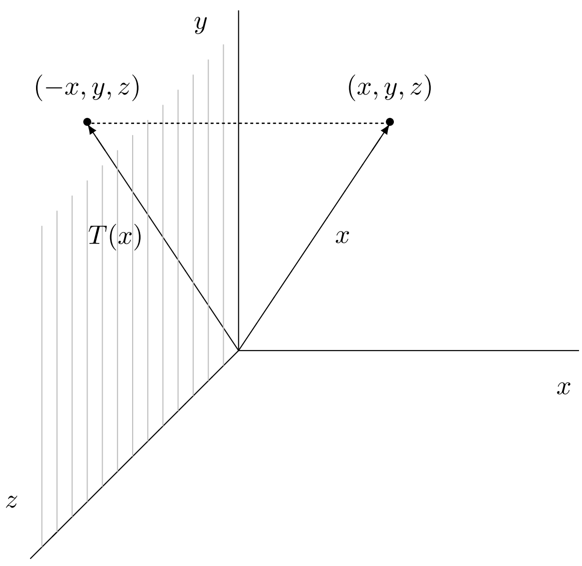
Step 2: Construct the matrix that has the images obtained in Step 1 as its successive columns. This matrix is the **standard matrix for the transformation.**

1.1 Reflection Operators

Some of the most basic matrix operators on $\mathbb{R}^3$ are those that map each point into its symmetric image about a fixed plane. These are called *reflection operators*. The following diagrams show the *standard matrices* for the reflections about the coordinate planes in $\mathbb{R}^3$. In each case the standard matrix was obtained by finding the images of the standard basis vectors, converting those images to column vectors, and then using those column vectors as successive column of the standard matrix.

The following matrices will perform **orthogonal reflections ABOUT** the xy–plane, the xz–plane and the yz–plane respectively.

Reflection about the yz–plane:
$T(x, y, z) = (-x, y, z)$



$T(e_1) = T(1, 0, 0) = (-1, 0, 0)$

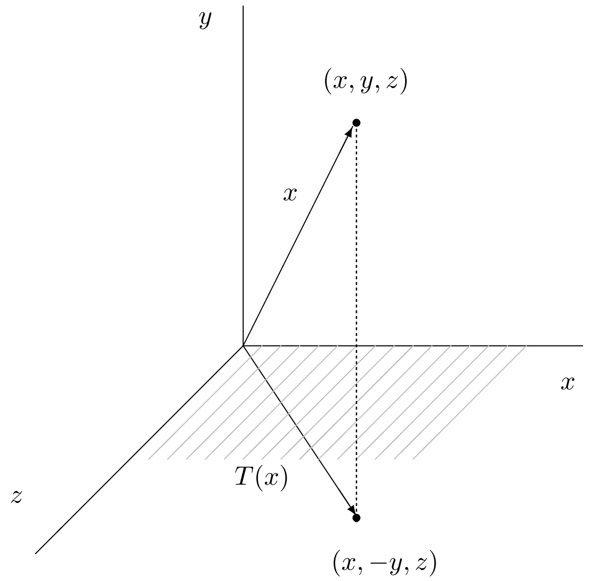
$T(e_2) = T(0, 1, 0) = (0, 1, 0)$

$T(e_3) = T(0, 0, 1) = (0, 0, 1)$

$$T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Reflection about the xz–plane:

$T(x, y, z) = (x, -y, z)$



$T(e_1) = T(1, 0, 0) = (1, 0, 0)$

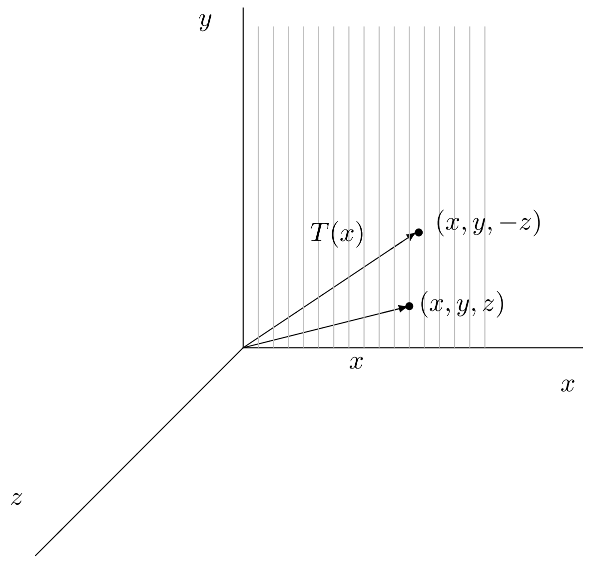
$T(e_2) = T(0, 1, 0) = (0, -1, 0)$

$T(e_3) = T(0, 0, 1) = (0, 0, 1)$

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Reflection about the xy–plane:

$T(x, y, z) = (x, y, -z)$



$T(e_1) = T(1, 0, 0) = (1, 0, 0)$

$T(e_2) = T(0, 1, 0) = (0, 1, 0)$

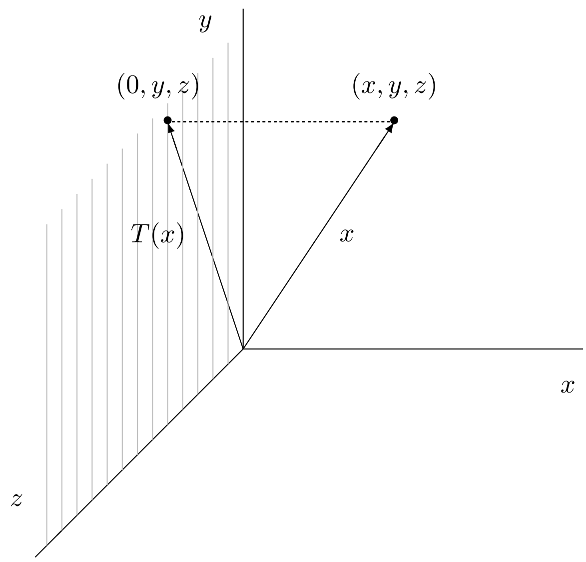
$T(e_3) = T(0, 0, 1) = (0, 0, -1)$

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

1.2 Projection Operators

Matrix operators on $\mathbb{R}^3$ that map each point into its orthogonal projection on a fixed plane are called *projection operators* (or more precisely, *orthogonal projection operators*). The following are the *standard matrices* for the orthogonal projections on the coordinate axes in $\mathbb{R}^3$. Again, in each case the standard matrix was obtained by finding the images of the standard basis vectors, converting those images to column vectors, and then using those column vectors as successive column of the standard matrix.

Orthogonal Projection onto the yz–plane:
$T(x, y, z) = (0, y, z)$



$$T(e_1) = T(1, 0, 0) = (0, 0, 0)$$
$$T(e_2) = T(0, 1, 0) = (0, 1, 0)$$
$$T(e_3) = T(0, 0, 1) = (0, 0, 1)$$

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The following matrices will perform **orthogonal projection ONTO** the xy–plane, the xz–plane and the yz–plane respectively.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example To determine the coordinates of the point $(1, 2, 3)$ after a reflection in the yz–plane, we have

$$T_A(x) = [T]x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

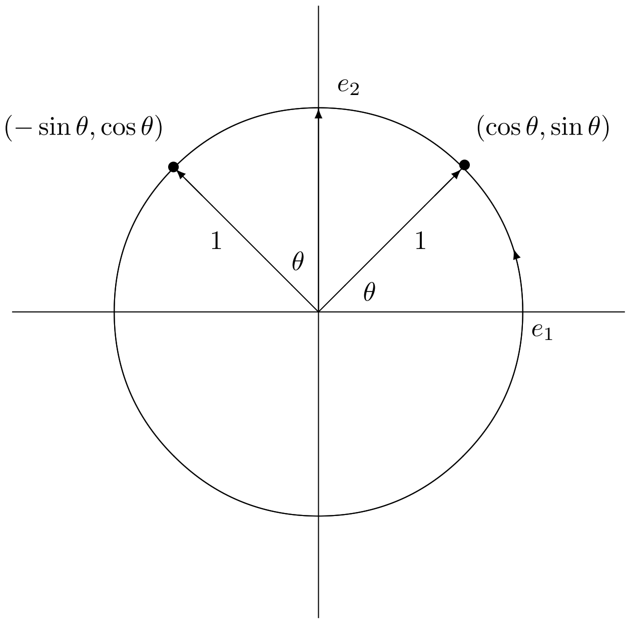
i.e., $(1, 2, 3) \longrightarrow (-1, 2, 3)$

1.3 Rotation Operators

Matrix Operators on on $\mathbb{R}^2$ and $\mathbb{R}^3$ that move points along circular arcs are called *rotation operators*. Let us consider how to find the standard matrix for the rotation operator

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

that moves points *anti-clockwise* about the origin through an angle θ.



The images of the standard basis vectors in $\mathbb{R}^2$ are

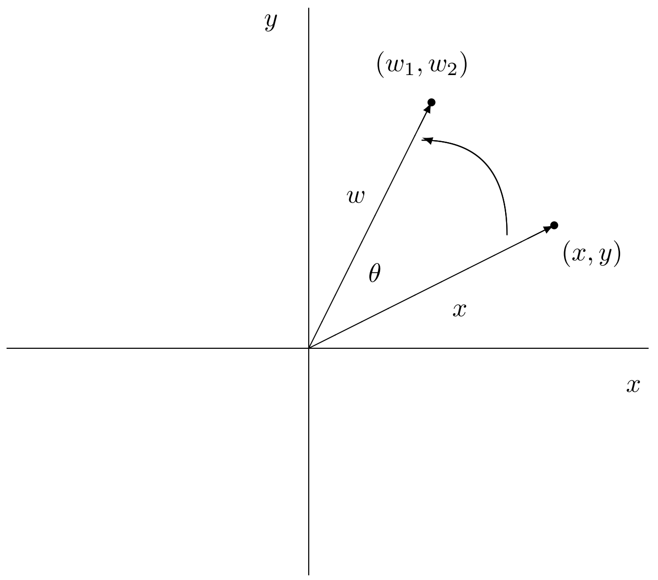
$$T(e_1) = T(1,0) = (\cos\theta, \sin\theta)$$
$$T(e_2) = T(0,1) = (-\sin\theta, \cos\theta)$$

Hence, the *standard matrix* for this transformation is

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

In keeping with common notation, we represent this operator by R_θ, i.e.,

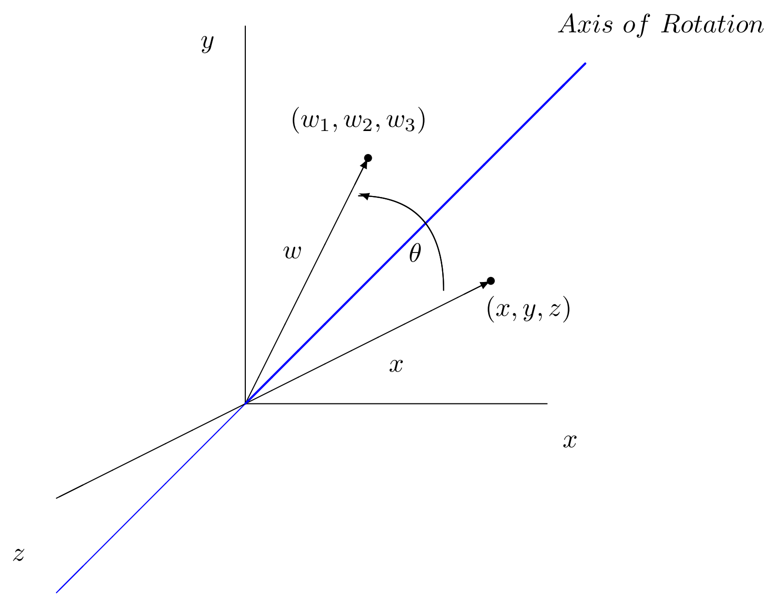
$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



Note In the plane, *anti-clockwise* angles are positive and *clockwise* angles are negative. The rotation matrix for a *clockwise* rotation of θ radians can be obtained by replacing θ by $-\theta$. Note that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. After simplification this yield

$$R_{-\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

A *rotational operator* in $\mathbb{R}^3$ is a matrix operator that rotates each vector in $\mathbb{R}^3$ about some rotation axis through fixed angle θ. A rotation of vectors in $\mathbb{R}^3$ is usually described in relation to a ray emanating from the origin, called the *axis of rotation.* As a vector revolves around the axis of rotation, it sweeps out some portion of a cone. The *angle of rotation*, which is measured in the base of the cone, is described as 'clockwise' or 'anti-clockwise' in relation to a viewpoint that is along the axis of rotation *looking towards the origin.*



The following three matrices will rotate any vector *anti-clockwise* about the x,y and z axes through an angle of θ respectively. For each of these rotations one of the components is unchanged and the relationship between the other components can be derived by the same procedure used to derive rotational matrices in $\mathbb{R}^2$. About the x-axis, y-axis and z-axis respectively, we have

$$R_x\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_y\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The following three matrices will rotate any vector *clockwise* about the x,y and z axes through an angle of θ respectively. About the x-axis, y-axis and z-axis respectively, we have

$$R_x\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

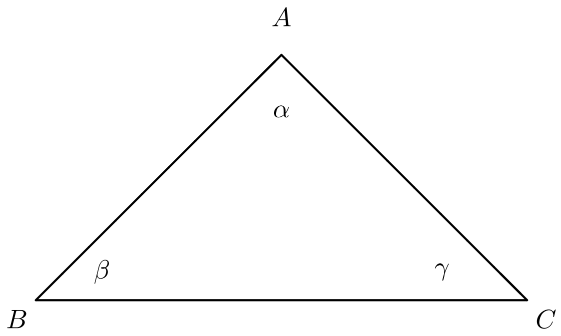
$$R_y\theta = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example Using a *matrix transformation* rotate the triangle ABC whose vertices are the points

$$A(1,-1,0) \quad , \quad B(2,1,-1) \quad , \quad C(-1,1,2)$$

through 60^o *anti-clockwise* about the z-axis as follows:



Firstly, the following matrix will rotate any vector *anti-clockwise* about the z-axes through an angle of 60^o

$$R_{60^o} = \begin{pmatrix} \cos 60^o & -\sin 60^o & 0 \\ \sin 60^o & \cos 60^o & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now

$$R_{60^o} = [T]x = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \cdot 366 \\ 0 \cdot 366 \\ 0 \end{pmatrix}$$

$$R_{60^o} = [T]x = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \cdot 134 \\ 2 \cdot 23 \\ -1 \end{pmatrix}$$

$$R_{60^o} = [T]x = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \cdot 366 \\ -0 \cdot 366 \\ 2 \end{pmatrix}$$

Finally, we have $A^{'}(1 \cdot 366, 0 \cdot 366, 0)$, $B^{'}(0 \cdot 134, 2 \cdot 23, -1)$, $C^{'}(-1 \cdot 366, -0 \cdot 366, 2)$.

Remark Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix represents a *clockwise* rotation about the z-axis through 90^o. Notice that the inverse matrix is

$$A^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is an *anti-clockwise* rotation about the z-axis through 90^o. It is easy to see here that the inverse matrix is simply the transpose of the original matrix A. This is very common in computer graphics and any matrix with this property is called an *orthogonal matrix*.

Definition A square matrix A is an *orthogonal* matrix if $A^{-1} = A^t$.

This can always be tested by multiplying $A.A^t$ and see if the result is the identity matrix. Notice the the three *clockwise* rotational matrices $R_x\theta$,$R_y\theta$ and $R_z\theta$ are each orthogonal since multiplying each matrix by its transpose will yield I the identity matrix.

So, for example,

$$R_x\theta.(R_x\theta)^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This can be confirmed for the matrix $R_y\theta$ and $R_z\theta$ in a similar way.

When an orthogonal matrix is used to rotate vectors, it will keep the lengths of the vectors preserved as will the angle between the vectors. So any orthogonal matrix may represent a rotation.

Example Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

To show that A is orthogonal, note that a square matrix A is an *orthogonal* matrix if $A^{-1} = A^t$. Pre-multiplying both sides by A yield

$$A.A^{-1} = I = A.A^t$$

Now

$$A.A^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1.4 Some Exercises

Exercise Using a *matrix transformation* rotate the triangle ABC whose vertices are the points

$$A(3, -4, 8) \quad , \quad B(2, 3, -5) \quad , \quad C(-1, 6, 7)$$

through 30^o *anti-clockwise* about the y-axis.

Exercise Using a *matrix transformation* rotate the cube whose corners are the points

$$A(1, 5, 1) \quad , \quad B(4, 5, 1) \quad , \quad C(1, 2, 1) \quad , \quad D(4, 2, 1)$$

$$E(1, 5, 4) \quad , \quad F(4, 5, 4) \quad , \quad G(1, 2, 4) \quad , \quad H(4, 2, 4)$$

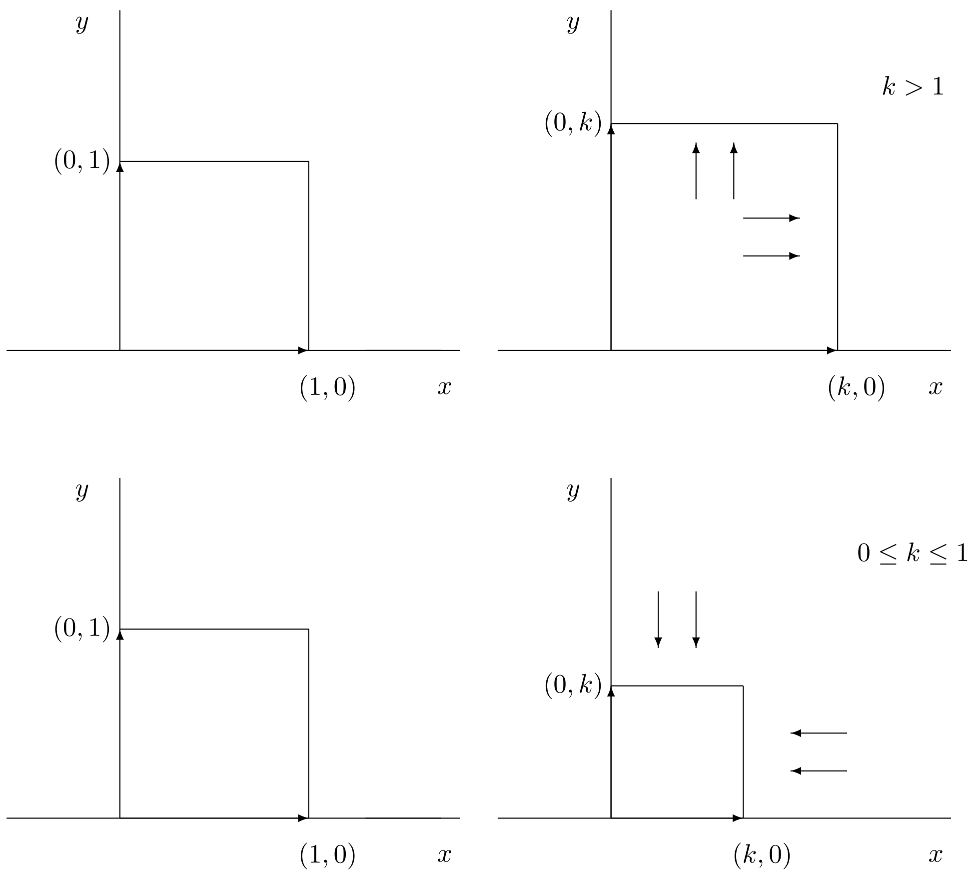
through 45^o *anti-clockwise* about the y-axis.

1.5 Dilations and Contractions

If k is a non-negative scalar, then the operator

$$T(x) = kx$$

in $\mathbb{R}^2$ or $\mathbb{R}^3$ has the effect of increasing or decreasing the length of each vector by a factor of k. If $0 \leq k \leq 1$ the operator is called a *contraction* with factor k, and if $k > 1$ it is called a *dilution* with factor k. If $k = 1$, then T is the identity operator and can be regarded as either a *contraction* or a *dilation*.



The following matrices will perform a *contraction* or a *dilation* with factor k in $\mathbb{R}^2$.

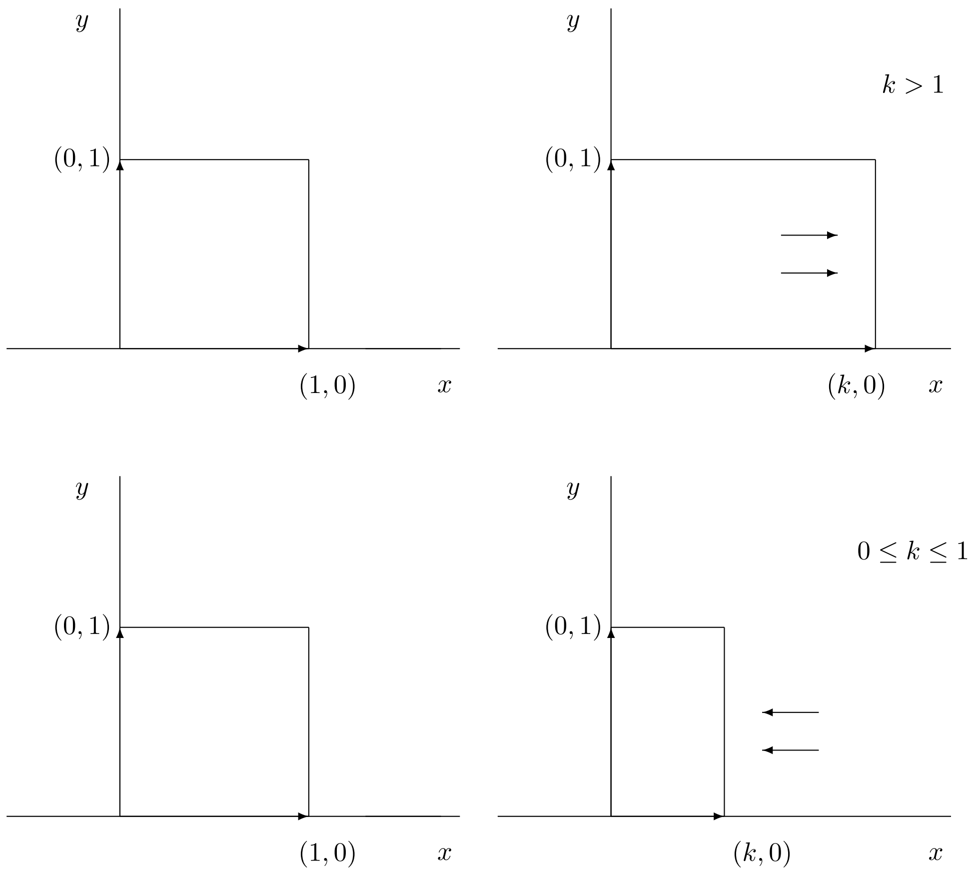
$$A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

The following matrices will perform a contraction or a dilation with factor k in $\mathbb{R}^3$.

$$A = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

In a dilation or contraction in $\mathbb{R}^2$ or $\mathbb{R}^3$, all coordinates are multiplied by a factor k. If only one of the coordinates is multiplied by by a factor k, then the resulting operator is called an *expansion* or *compression* with factor k.

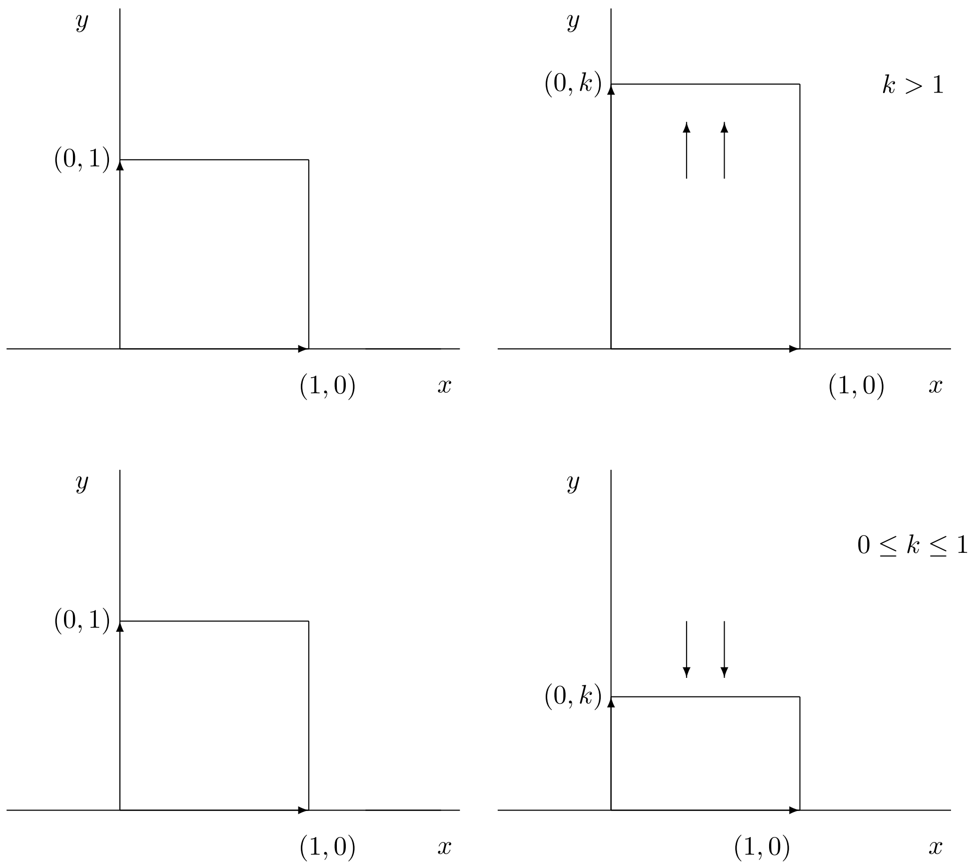
Consider an *expansion* or *compression* with factor k in the x-direction.



The following matrices will perform a *expansion* or *compression* with factor k in $\mathbb{R}^2$ in the x-direction.

$$A = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$$

Consider an *expansion* or *compression* with factor k in the y-direction.



The following matrices will perform a *expansion* or *compression* with factor k in $\mathbb{R}^2$ in the y-direction.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$$

The following matrices will perform a *expansion* or *compression* with factor k in $\mathbb{R}^3$ in the x-direction, in the y-direction and in the z-direction respectively.

$$A = \begin{pmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{pmatrix}$$

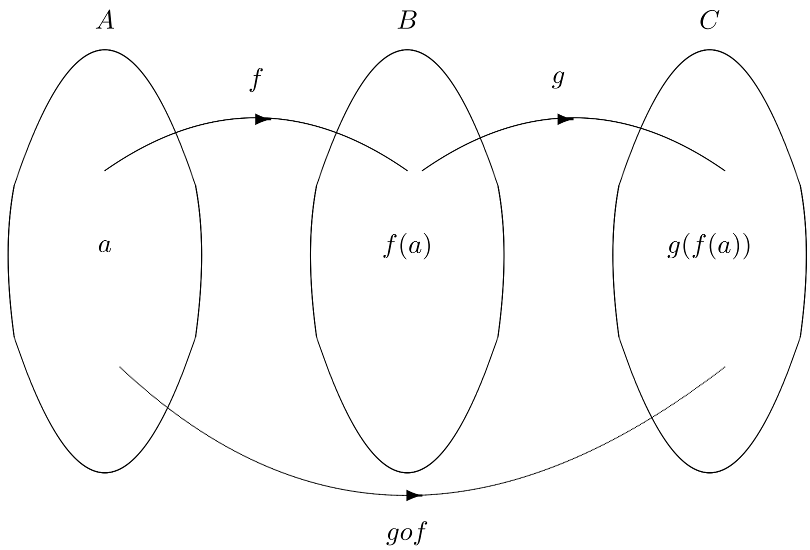
1.6 Properties of Matrix Transformations

We seek to show that if several matrix transformations are performed in succession, then the same result can be obtained by a single matrix transformation that is chosen appropriately. Firstly, we recall the definition of a composite function.

Let A, B and C be sets, let $f : A \longrightarrow B$ be a function A to B, let $g : B \longrightarrow C$ be a function from B to C. Then there is a function $g \circ f : A \longrightarrow C$ obtained by composing the functions f and g. This function is defined at each element a of A by the formula

$$(g \circ f)(a) = g(f(a))$$

In other words, in order to apply the composition function $g \circ f$ to an element a of A, we first apply the function f to the element a, and then we apply the function g to the resulting element $f(a)$ of B to obtain an element $g(f(a))$ of C.



Remark Note that '$g \circ f$' denotes the composition function 'f followed by g'. The functions are specified in this order (which may at first seem odd) in order that $(g \circ f)(a) = g(f(a))$ for all elements a of the domain A of the function f.

Example Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$ and let $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = x + 5$. Now $(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 5$. Also $(f \circ g)(x) = f(g(x)) = f(x + 5) = x^2 + 10x + 25$

Exercise Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = (x + 1)^2$ and let $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = \sin x$. Determine the composite functions $g \circ f$ and $f \circ g$.

Now, consider the following matrix transformations

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

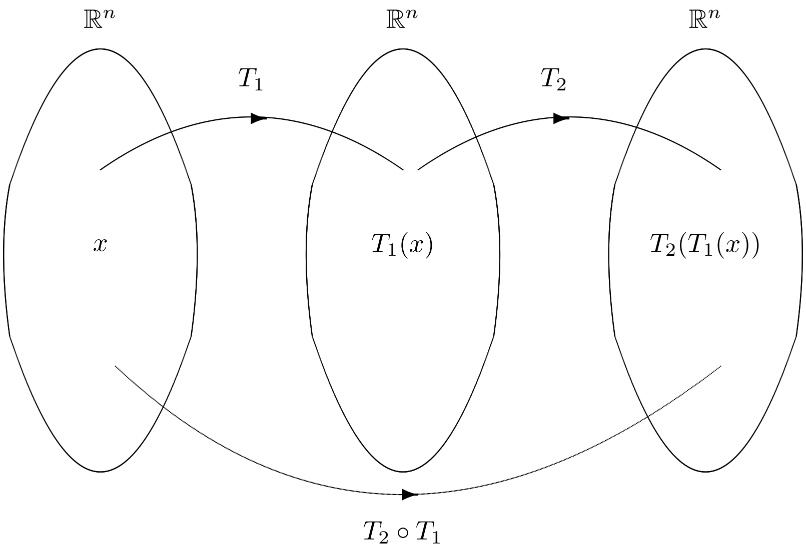
$$T_B : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

The composition of T_A with T_B will be denoted by

$$T_B \circ T_A$$

Note that '$T_B \circ T_A$' denotes the composition transformation 'T_A followed by T_B', i.e., the transformation T_A is performed first. This composition of transformations is defined as

$$(T_B \circ T_A)(x) = [T_B][T_A]$$



Example Let $T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that

$$T(x) = Ax$$

denote a *matrix transformation* with *standard matrix* A. To find the *standard matrix* for the operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that first reflects about the line $y = x$, then rotates through an angle of 180^o *anti-clockwise* about the origin, we proceed as follows. Firstly, the standard matrix A for the operator can be expressed as the composition

$$A = T_2 \circ T_1$$

where T_1 is the reflection about the line $y = x$ and T_2 rotates through an angle of 180^o *anti-clockwise* about the origin. The *standard matrices* for these operations are

$$[T_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$[T_2] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

thus, it follows, that the *standard matrix* $A = T_2 \circ T_1$ is

$$[T_2][T_1] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Therefore

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Example Let $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that

$$T(x) = Ax$$

denote a *matrix transformation* with *standard matrix* A. To find the *standard matrix* for the operator $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that first rotates a vector *anti-clockwise* about the z-axis through an angle θ, then reflects the resulting vector about the xz-plane, and then projects that vector orthogonally onto the xy-plane, we proceed as follows. Firstly, the standard matrix A for the operator can be expressed as the composition

$$A = T_3 \circ T_2 \circ T_1$$

where T_1 is the rotation about the z-axis, T_2 is the reflection about the xz-plane, and T_3 is the orthogonal projection n the xy-plane. The *standard matrices* for these operations are

$$[T_1] = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[T_2] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[T_3] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

thus, it follows, that the *standard matrix* $A = T_3 \circ T_2 \circ T_1$ is

$$[T_3][T_2][T_1] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ -\sin\theta & -\cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example Let $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that

$$T(x) = Ax$$

denote a *matrix transformation* with *standard matrix* A. To find the *standard matrix* for the operator $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that first rotates a vector *anti-clockwise* about the x-axis through an angle 30^o, then rotates the resulting vector *anti-clockwise* about the z-axis through an angle of 30^o, and then contracts that vector by a factor of $k = \frac{1}{4}$, we proceed as follows. Firstly, the standard matrix A for the operator can be expressed as the composition

$$A = T_3 \circ T_2 \circ T_1$$

where T_1 is the rotation *anti-clockwise* about the x-axis through an angle 30^o, T_2 is the rotation *anti-clockwise* about the z-axis through an angle 30^o, and T_3 contraction by a factor of $k = \frac{1}{4}$. The *standard matrices* for these operations are

$$[T_1] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$[T_2] = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[T_3] = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

thus, it follows, that the *standard matrix* $A = T_3 \circ T_2 \circ T_1$ is

$$[T_3][T_2][T_1] = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Therefore

$$A = \begin{pmatrix} \frac{\sqrt{3}}{8} & -\frac{\sqrt{3}}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{3}{16} & -\frac{\sqrt{3}}{16} \\ 0 & \frac{1}{8} & \frac{\sqrt{3}}{8} \end{pmatrix}$$

Exercise Let $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that

$$T(x) = Ax$$

denote a *matrix transformation* with *standard matrix* A. Find the *standard matrix* for the stated composition in $\mathbb{R}^3$.

i A reflection about the xy-plane, followed by a reflection about the xz-plane, followed by an orthogonal projection on the yz-plane.

ii A rotation of 270^o *anti-clockwise* about the x-axis, followed by a rotation of 90^o *anti-clockwise* about the y-axis, followed by a rotation of 180^o *anti-clockwise* about the z-axis.