INSTITIÚID TEICNEOLAÍOCHTA CHEATHARLACH

INSTITUTE OF TECHNOLOGY CARLOW

FUNCTIONS AND THEIR GRAPHS

1 Functions and their Graphs

1.1 Some Important Sets in Mathematics

We define the following important sets.
A *natural number* is a positive whole number.

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, ...\}$$

An *integer* is a positive or negative whole number (including 0).

$$\mathbb{Z} = \{.......... -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$$

A *rational* or *quotient* number is a number that can be written in the form $\frac{a}{b}$, where a and b are integers with $b \neq 0$. The set of rational numbers is denoted by $\mathbb{Q}$.
A *real* number is rational if and only if, when expressed as a decimal, it has a finite or *recurring* expansion. For example,

$$\frac{5}{4} = 1.25, \quad \frac{2}{3} = 0.\dot{6} \quad \frac{20}{7} = 2.85714\dot{2}$$

A famous proof, attributed to Pythagoras, shows that $\sqrt{2}$ is not rational, and e and π are also known to be *irrational*. A *real number* is a number used in mathematics, in scientific work and in everyday life. The set of real numbers is denoted by $\mathbb{R}$ and contains numbers such as $0, \frac{1}{2}, -2, 4, 75, \sqrt{2}$ and π. There is no real number x such that $x^2 + 1 = 0$. The introduction of a new number i such that $i^2 = -1$ gives rise to further numbers of the form $a + ib$. A number of the form $a + ib$, where a and b are real, is a *complex number*. The set of complex numbers is denoted by $\mathbb{C}$. A *prime number* is a number greater than 1 that has only two positive divisors - itself and one.

$$\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, 19, ...\}$$

1.2 Cartesian Product of Sets

Let A and B be sets. The *Cartesian product* $A \times B$ of the sets A and B is defined to be the set of all *ordered pairs* (a, b) with $a \in A$ and $b \in B$.

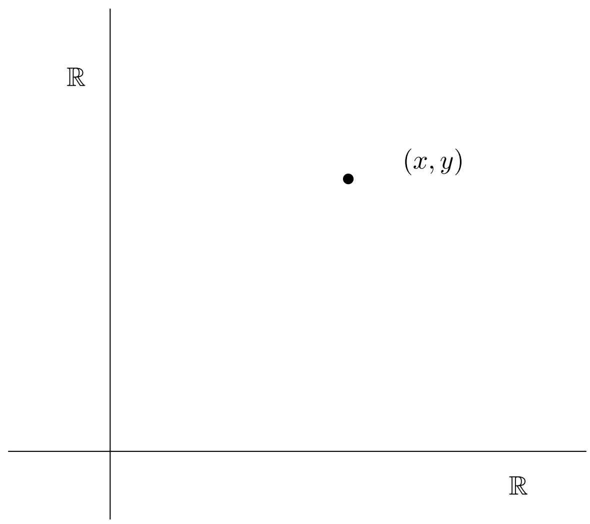
Such an ordered pair (a, b) is comprised of two elements a and b, where the first element a is taken from the set A, and the second element b is taken from the the set B. If (a_1, b_1) and (a_2, b_2) are ordered pairs of this type then $(a_1, b_1) = (a_2, b_2)$ if and only if $a_1 = a_2$ and $b_1 = b_2$.

Example Let $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Then

$$A \times B = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\}$$

Suppose A and B are finite sets. Let m and n be the number of elements in A and B respectively. Then the number of elements of the Cartesian product $A \times B$ is mn. (There are m ways to choose the element a from A, and, for each choice, there are n ways to choose the element b from B.)

Example Points of the plane are specified in Cartesian coordinates by means of ordered pairs (x, y), where x and y are real numbers. The set of such ordered pairs is the set $\mathbb{R} \times \mathbb{R}$.



The definition of Cartesian product may be extended to more than two sets.

Definition Let $A_1, A_2, ...A_n$ be sets. The Cartesian product of these sets is the set $A_1 \times A_2 \times ....\times A_n$ consisting of all ordered n-tuples $(a_1, a_2, ...a_n)$ with $a_i \in A_i$ for $i = 1, 2, ...n$.

Example Let $A = \{1, 2, 3\}$, $B = \{a, b\}$ and $C = \{\pi\}$. Then

$$A \times B \times C = \{(1, a, \pi), (1, b, \pi), (2, a, \pi), (2, b, \pi), (3, a, \pi), (3, b, \pi)\}$$

Note, that in the previous example that the sets A, B and C have $3, 2$ and 1 elements respectively, then their Cartesian product have 6 elements, since $6 = 3 \times 2 \times 1$.

Example Points of three dimensional space are specified in Cartesian coordinates by means of ordered triples (x, y, z), where x, y and z are real numbers. The set of such ordered triples is the set $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Example Suppose we wish to construct a database containing information on students taking courses at I.T. Carlow. Each record in the database is to specify the student number, the name, and the course being followed by the student. Let

$$\begin{aligned} I &= \textit{set of all strings of eight decimal digits} \\ N &= \textit{set of all student names} \\ D &= \textit{set of all courses taught at I.T. Carlow.} \end{aligned}$$

Then a record in the database is an element of the set $I \times N \times D$, such as

$$(54344019, \textit{James Byrne}, CW046)$$

The collection of all such records contained in the database can be viewed as a subset of the Cartesian product $I \times N \times D$ of the set I, N and D. The language of sets and Cartesian products is used in discussions of *relational databases.*

Exercise Let $A = \{1, 2\}$, $B = \{a, b\}$ and $C = \{\alpha, \beta\}$. Determine each of the following:

$$A \times B \qquad B \times A \qquad A \times B \times C \qquad B \times B \times B.$$

A subset of the Cartesian product of two sets is of special importance.

1.3 Binary Relations

A *binary relation* on a set specifies relations between pairs of elements from the set.

Example The relations $=$ ('equals'), $\neq$ ('not equal to'), $<$ ('less than'), $>$ ('greater than'), $\leqslant$ ('less than or equal to'), $\geqslant$ ('greater than or equal to') are all binary relations on the set $\mathbb{R}$ of real numbers.

More formally we have the following definition.

Definition Let A be a set. A *binary relation* R on A is a subset of the Cartesian product $A \times A$, i.e.,

$$R = \{(a, b) \in A \times A : aRb\}$$

If an ordered pair (a, b) is an element of R, then we say that 'a is related to b' and we write aRb.

Example Let $A = \{1, 2\}$. A binary relation R on A may be defined as follows:

$$R = \{(a, b) \in A \times A : a \leqslant b\}.$$

To write R as a list of elements, firstly $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Hence $R = \{(1, 1), (1, 2), (2, 2)\}$.

Example Let A be a set, and let $P(A)$ be the power set of A (i.e., the set whose elements are the subsets of A). Then $\subset$ is a binary relation on $P(A)$, where two subsets B and C of A satisfy $B \subset C$ if and only if B is a subset of C.

Exercise Let $A = \{1\}$. A binary relation R on $P(A)$, the power set of A, may be defined as follows:

$$R = \{(a, b) \in P(A) \times P(A) : a \subset b\}.$$

List the elements of R.

Definition Let R be a relation on a set A.

The relation R is said to be *reflexive* when it has the following property: xRx for all elements x of the set A.

The relation R is said to be *symmetric* when it has the following property: if x and y are elements of the set A, and if xRy, then yRx.

The relation R is said to be *transitive* when it has the following property: if x, y and z are elements of the set A, and if xRy and yRz, then xRz.

An *equivalence relation* is a relation that is reflexive, symmetric and transitive.

Example The relation $R = \{(1,1),(1,3),(1,5),(2,2),(2,4),(3,1),(3,3),(3,5),$
$(4,2),(4,4),(5,1),(5,3),(5,5)\}$ on the set $A = \{1,2,3,4,5\}$ is an equivalence relation. The relation R is reflexive since $(1,1),(2,2),(3,3),(4,4),(5,5) \in R$. The relation R is symmetric since if (x,y) is in R, then (y,x) is in R. The relation R is transitive since if (x,y) and (y,z) are in R, then (x,z) is in R.

Example The relation $<$ ('less than') on the set $\mathbb{R}$ of real numbers is neither reflexive nor symmetric, but is transitive.

$$R = \{(a,b) \in \mathbb{R} \times \mathbb{R} : a < b\}$$

The relation R is not reflexive since there is no real number x satisfying $x < x$. The relation R is not symmetric since there is no real numbers x and y satisfying both $x < y$ and $y < x$. However, if x, y and z are real numbers, and if $x < y$ and $y < z$, then $x < z$, and therefore the relation $<$ on $\mathbb{R}$ is transitive.

Example The relation $=$ ('equals') on the set $\mathbb{R}$ of real numbers is an equivalence relation.

$$R = \{(a,b) \in \mathbb{R} \times \mathbb{R} : a = b\}$$

The relation R is reflexive since $x = x$ for all real numbers x. The relation R is symmetric since if x and y are real numbers, and if $x = y$, then $y = x$. Also, the relation R is transitive since if x, y and z are real numbers, and if $x = y$ and $y = z$, then $x = z$.

Example None of the relations $\neq$ ('not equal to'), $<$ ('less than'), $>$ ('greater than'), $\leqslant$ ('less than or equal to'), $\geqslant$ ('greater than or equal to') are equivalence relations on the set $\mathbb{R}$ of real numbers.

Example Let A be a non-empty set, and let $P(A)$ be the power set of A. The relation $\subset$ on $P(A)$ is reflexive and transitive but not symmetric.

$$R = \{(a, b) \in P(A) \times P(A) : a \subset b\}$$

Let B, C and D be subsets of A. The relation R is reflexive since every subset of A is a subset of itself. The relation R is not symmetric since if $B \subset C$, then it is not the case that $C \subset B$. The relation R is transitive since if $B \subset C$ and $C \subset D$, then $B \subset D$.

A subset of the Cartesian product of n sets $A_1, A_2, ..., A_n$ is sometimes referred to as a *n-ary relation* on the sets $A_1, A_2, ..., A_n$ and is denoted by $A_1 \times A_2 \times .... \times A_n$.

1.4 Functions

Definition Let A and B be sets. A *function* $f : A \longrightarrow B$ from A to B assigns to each element a of A a (unique) element $f(a)$ of B. The set A on which the function is defined is referred to as the *domain* of the function $f : A \longrightarrow B$. The set B into which the domain is mapped by f is referred to as the *codomain* of the function f.

In other words, a *function* (or mapping) from A into B is a relation from A to B, in which no two distinct ordered pairs have the same first element.

$$f = \{(x, y) \in A \times B : y = f(x)\}$$

Definition The first variable x in the ordered pair (x, y) is often called the *independent variable* of the function f. The second variable y is called the *dependent variable.*

Definition A function f whose domain and range are sets of real numbers is said to be a *real valued function.*

Example Let $\mathbb{R}$ be the set of real numbers. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$ for all real numbers x is a function from the set of real numbers to itself.

$$f = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2\}$$

Example There is a function $f : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$, defined by $f(x) = 1/x$ for all non-zero real numbers x. The *domain* of this function is the set $\mathbb{R}\setminus\{0\}$ of all non-zero real numbers. (i.e., the set $\{x \in \mathbb{R} : x \neq 0\}$). The domain of this function cannot be extended to the entire set $\mathbb{R}$ of real numbers since the reciprocal of zero is not defined.

Example Let A be the set of letters in the English alphabet (including both upper-case and lower-case letters). Then there is a function $f : A \longrightarrow \mathbb{N}$ which sends each letter to its ASCII code. Then, for example, $f(A) = 65, f(B) = 66, f(a) = 97$ and $f(b) = 98$.

Definition Let A and B be sets and let $f : A \longrightarrow B$ be a function from A to B. The *range* of the function f is the subset $f(A)$ of B defined by

$$f(A) = \{b \in B : b = f(a) \ for \ some \ a \in A\}$$

In other words, the *range* of a function is the set consisting of all elements of the codomain of the function that are images under the function.

Finally, the definition of the identity function:

Definition The function $I_A : A \longrightarrow A$ from the set A to itself is a function which sends each element a of A to itself. This function is referred to as the *identity function*.

The identity function $I_A : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f(x) = x$.

1.5 Composite Functions

Let A, B and C be sets, let $f : A \longrightarrow B$ be a function A to B, let $g : B \longrightarrow C$ be a function from B to C. Then there is a function $g \circ f : A \longrightarrow C$ obtained by composing the functions f and g. This function is defined at each element a of A by the formula

$$(g \circ f)(a) = g(f(a))$$

In other words, in order to apply the composition function $g \circ f$ to an element a of A, we first apply the function f to the element a, and then we apply the function g to the resulting element $f(a)$ of B to obtain an element $g(f(a))$ of C.

Remark Note that '$g \circ f$' denotes the composition function 'f followed by g'. The functions are specified in this order (which may at first seem odd) in order that $(g \circ f)(a) = g(f(a))$ for all elements a of the domain A of the function f.

Example Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, $C = \{w, x, y, z\}$. Now let $f : A \longrightarrow B$ such that $f = \{(1, a), (2, a), (3, b), (4, c)\}$ and $g : B \longrightarrow C$ such that $g = \{(a, x), (b, y), (c, z)\}$. Now $g \circ f : A \longrightarrow C$ with $g \circ f = \{(1, x), (2, x), (3, y), (4, z)\}$.

Example Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$ and let $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = x + 5$. Now $(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 5$. Also $(f \circ g)(x) = f(g(x)) = f(x + 5) = x^2 + 10x + 25$

Exercise Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = (x + 1)^2$ and let $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = \sin x$. Determine the composite functions $g \circ f$ and $f \circ g$.

1.6 Graph of a Function

Let A and B be sets. To every function $f : A \longrightarrow B$ from A to B there corresponds a subset

$$\Psi(f) = \{(a, b) \in A \times B : b = f(a)\}$$

Mathematicians often refer to the subset of $A \times B$ corresponding to a function $f : A \longrightarrow B$ as the *graph* of the function. This subset consists of the Cartesian coordinates of the points of the plane that lie on the curve that represents the graph of the given function. Rather than considering a mixture of functions we will look at the graphs and characteristics of some important classes of functions, namely polynomial, exponential and logarithmic functions.

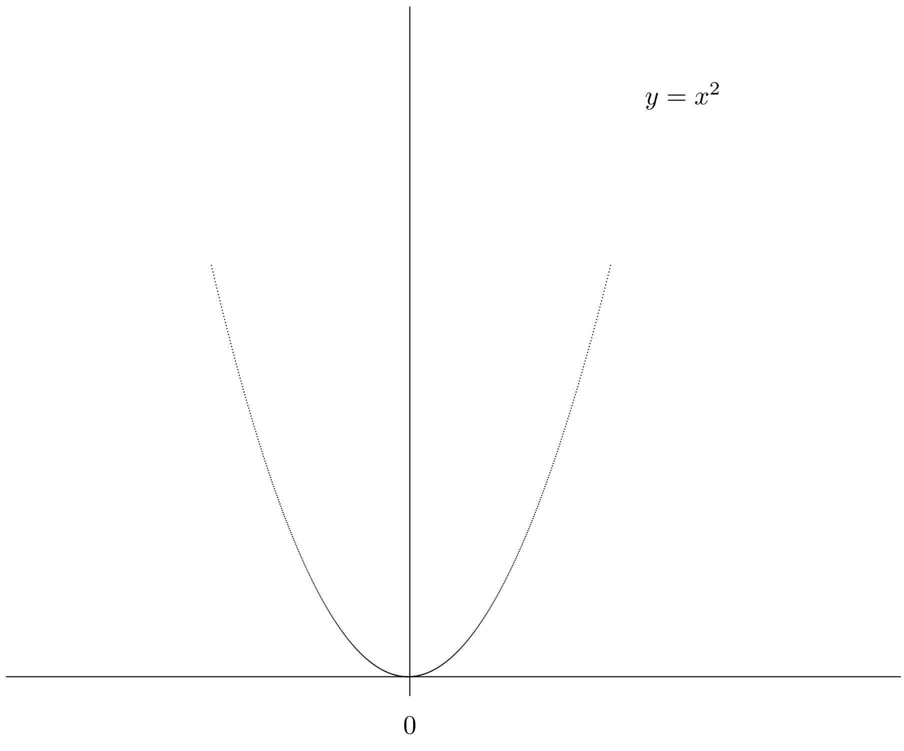
1.7 Polynomial Functions

Definition A function of the form

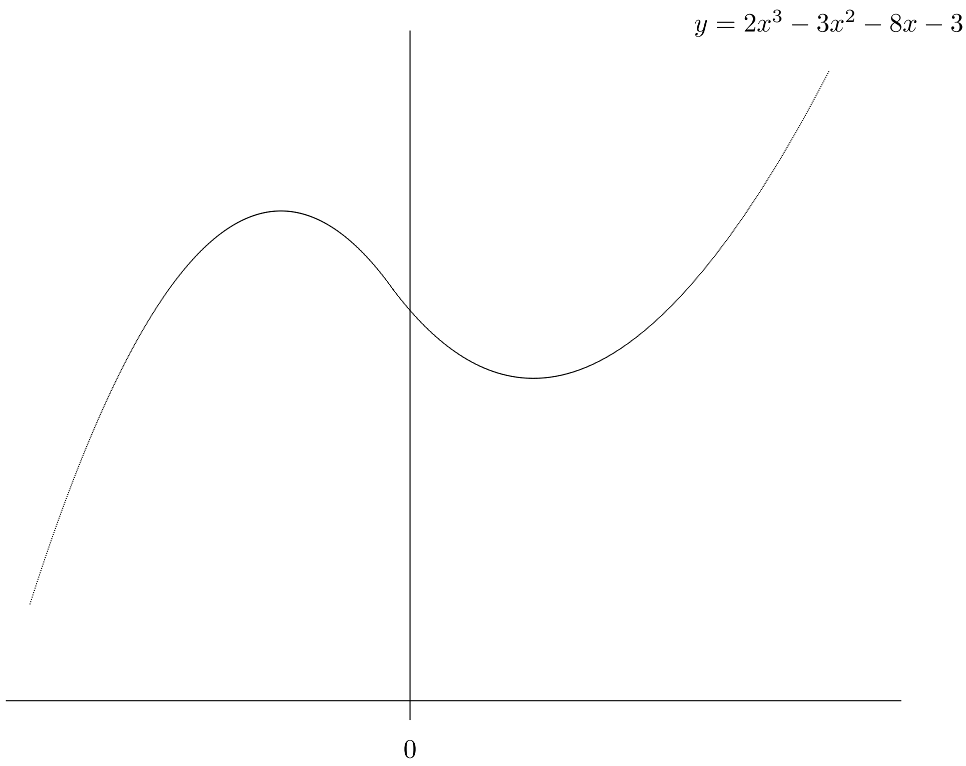
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_o$$

with $a_n \neq 0$ where n is a non-negative integer is called a *polynomial function* of degree n. The coefficients of the polynomial are $a_n, a_{n-1}, ..., a_2, a_1, a_0 \in \mathbb{R}$. polynomials may be described as linear, quadratic, cubic, quartic, quintic etc., according to their degree.

Example Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$. This function is defined for all real values of x. For every real $x \in \mathbb{R}$, the value $f(x)$ is real, hence it is a real valued function. We have $f(0) = 0, f(-1) = f(1) = 1$ and $f(-2) = f(2) = 4$. The graph of this function is given below.



Example Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = 2x^3 - 3x^2 - 8x - 3$. This function is defined for all real values of x. For every real $x \in \mathbb{R}$, the value $f(x)$ is real, hence it is a real valued function. We have $f(-2) = -15, f(-1) = 0, f(0) = -3, f(1) = -12, f(2) = -15$ and $f(3) = 0$. The graph of this function is given below.



1.8 Exponential Functions

Definition The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = a.b^{kx}$$

where $a, b, k \in \mathbb{R}$, $b > 1$, $a > 0$ is referred to as an *exponential function.*

Remark We must have $b > 1$. If $b < 1$ we would obtain complex numbers which we would like to avoid. If $b = 1$, the function reduces to $f(x) = a$ which is just a constant function. If $0 < b < 1$, we can arrange using the laws of indices to bring it to the required form. So for example

$$f(x) = \left(\frac{2}{3}\right)^x = \frac{2^x}{3^x} = \frac{3^{-x}}{2^{-x}} = \left(\frac{3}{2}\right)^{-x}$$

Remark For an exponential function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = a.b^{kx}$$

where $a, b, k \in \mathbb{R}$, $b > 1$, $a > 0$ we have the following properties

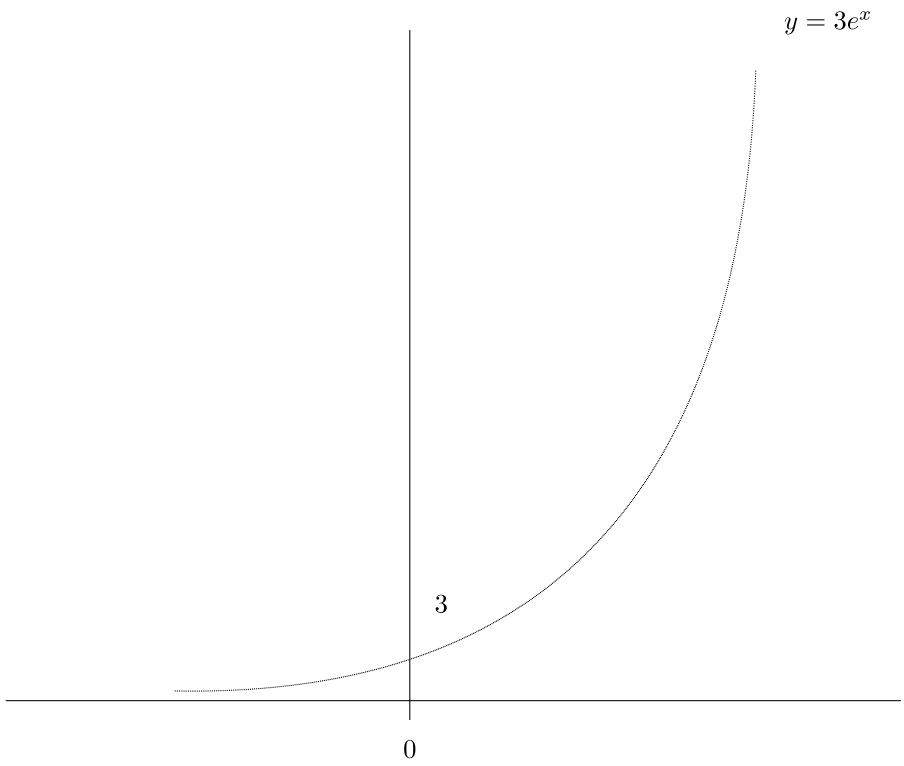
1. The y-intercept of $f(x)$ is a.
2. If $k > 0$, the negative x-axis is the horizontal asymptote.
3. If $k < 0$, the positive x-axis is the horizontal asymptote.
4. If k and a have the same sign, $f(x)$ is an increasing function.
5. If k and a have opposite signs, $f(x)$ is a decreasing function.

Definition An irrational number denoted by the letter e is defined by the following limit

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

The value of this constant is approximately 2.7182818. This constant forms the base of many important exponential functions.

Example Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = 3e^x$. For every $x \in \mathbb{R}$, the value $f(x)$ is real. We have $f(-3) = 0.14936, f(-2) = 0.4060, f(-1) = 1.1036, f(0) = 3, f(1) = 8.1548, f(2) = 22.167$ and $f(3) = 60.2566$. The graph of this function is given below.



1.9 Logarithmic Functions

Before giving the definition of a logarithmic function we first define a logarithm.

Definition Let b denote a positive real number other than 1. If $x \in \mathbb{R}_0^+$ we write $\log_b x$ to denote the logarithm of x to base b, which is the (unique) real number y which satisfies $b^y = x$.

In other words $\log_b x$ is the power (or exponent) to which we raise the base b in order to obtain x. Hence

$$\begin{array}{rcl} & y = & \log_b x \\ \Leftrightarrow & x = & b^y \end{array}$$

Example If $10^x = 3$, then $x = \log_{10} 3 = 0.4771$.

Also, if $\log_{10} x = 4$, then $x = 10^4 = 10,000$.

Remark Let b denote a positive real number other than 1. It follows that $\log_b b = 1$ since $b^1 = b$. There are two logarithm function on a scientific calculator – the *common logarithm*, denoted by log, has base 10 and the *natural logarithm*, denoted by ln, has base e, the exponential constant. However, to evaluate say $\log_8 9$ we require the following change of base formula.

$$\log_a x = \frac{\log_b x}{\log_b a}$$

So, for example

$$\log_8 9 = \frac{\log_{10} 9}{\log_{10} 8} = \frac{0.9542}{0.9030} = 1.056$$

Now for the definition of a logarithmic function.

Definition The function $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ defined by

$$f(x) = \log_b x$$

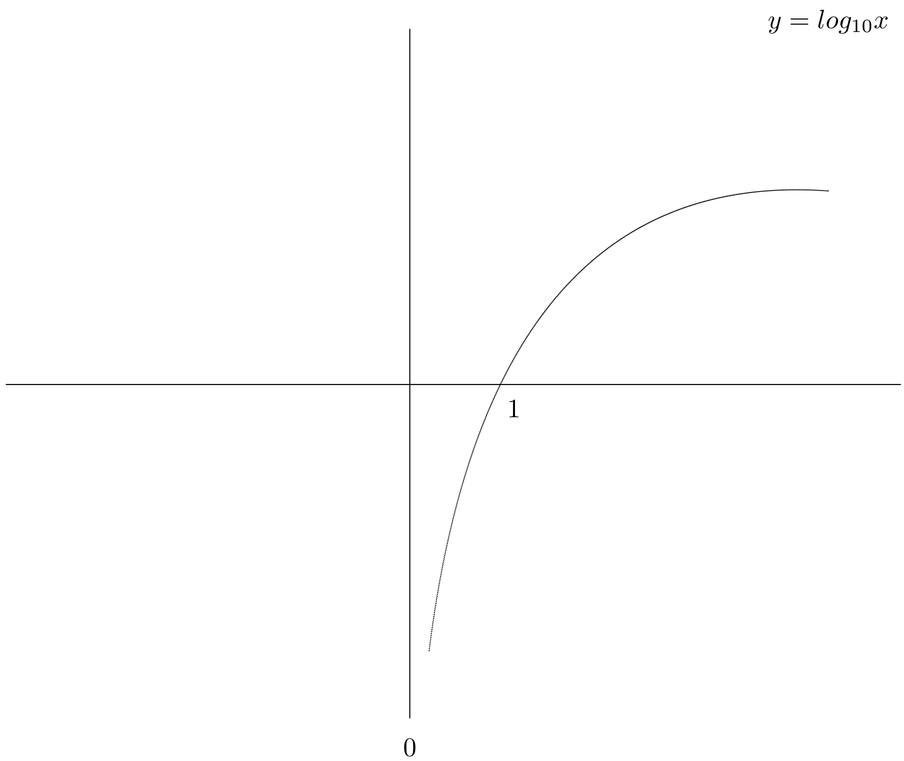
where b is a positive real number other than 1, is referred to as a *logarithmic function*.

Remark A logarithmic function will only map $\mathbb{R}_0^+ \longrightarrow \mathbb{R}$. It is easy to see this when you consider the definition of a logarithm. Recall that

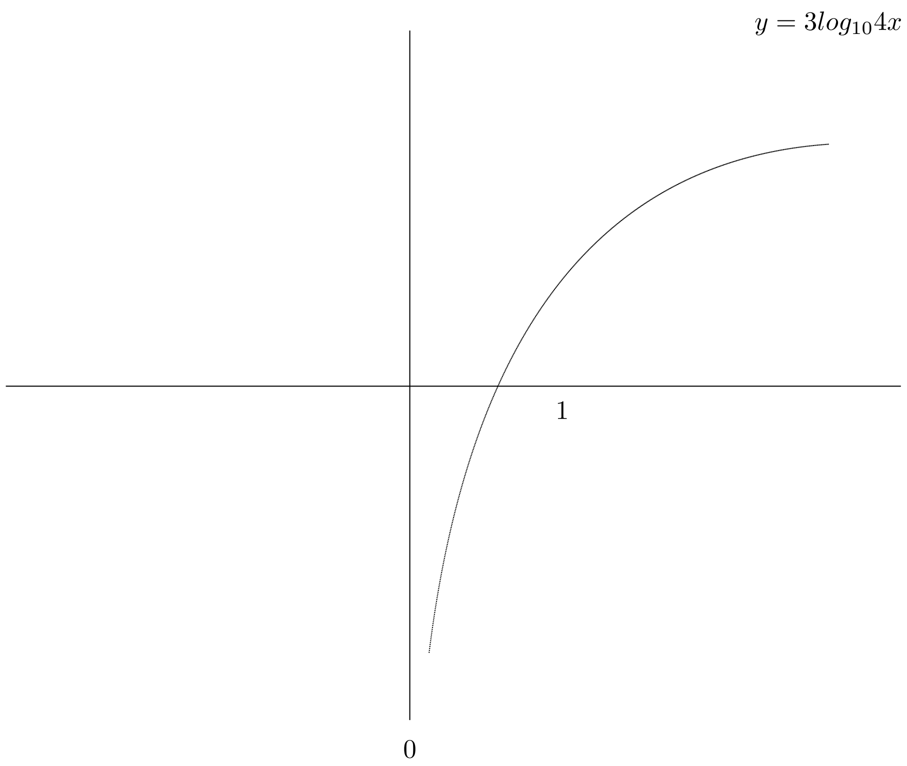
$$\begin{array}{rrcl} & y & = & \log_b x \\ \Leftrightarrow & x & = & b^y \end{array}$$

The value of b^y is always positive – if b is a positive real number other than 1 and y takes on either a positive or negative value it is clear that b^y is always positive and can never be zero. Hence x must be an element of $\mathbb{R}_0^+$.

Example Consider the function $f : \mathbb{R}^+ \setminus \{0\} \longrightarrow \mathbb{R}$ defined by $f(x) = log_{10}x$. This function is defined for all non-negative real values of x but not defined for any negative values of x or for $x = 0$. For every $x \in \mathbb{R}^+ \setminus \{0\}$, the value $f(x)$ is real. We have $f(0.01) = -2, f(0.1) = 1, f(0.15) = -0.824, f(0.2) = -0.698, f(0.8) = -0.097, f(1) = 0, f(2) = 0.3, f(5) = 0.698$ and $f(10) = 1$. The graph of this function is given below.



Example Let $f : \mathbb{R}^+ \setminus \{0\} \longrightarrow \mathbb{R}$ defined by $f(x) = 3\log_{10} 4x$. For every $x \in \mathbb{R}$, the value $f(x)$ is real. We have $f(0.01) = -4.1938, f(0.1) = -1.1938, f(0.15) = -0.6655, f(0.2) = -0.2907, f(0.8) = 1.5154, f(1) = 1.8062, f(2) = 2.709, f(5) = 3.903$ and $f(10) = 4.8062$. The graph of this function is given below.



Remark For a logarithmic function $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ defined by $f(x) = \log_b x$ where b is a positive real number other than 1, we have the following properties

1. If $b > 1$, $f(x) \geq 0, \forall x \geq 1$ and $f(x) < 0, \forall x < 1$.
2. If $0 < b < 1$, $f(x) \geq 0, \forall x \leq 1$ and $f(x) < 0, \forall x > 1$.
3. For the logarithmic function $f(x) = \log_b x$, $f(1) = 0$.

1.10 Inverse of a Function

Recall that the identity function $I_A : A \longrightarrow A$ from the set A to itself is a function which sends each element a of A to itself. With this in mind we can give the definition of an inverse function.

Definition Let A and B be sets. A function $f : A \longrightarrow B$ from A to B is said to be *invertible* if there exists a function $g : B \longrightarrow A$ from B to A such that

$$g \circ f = I_A$$

$$f \circ g = I_B$$

This function g, if it exists, is unique and is called the *inverse* of f and is denoted by f^{-1}

Not all function are invertible or in other words not all function have an inverse. In order for a function to have an inverse it must have the properties of being both *injective* and *surjective*.

Definition A function $f : A \longrightarrow B$ is said to be *injective* (or one-to-one) if $f(u) \neq f(v)$ whenever u and v are elements of the domain A with $u \neq v$.

Definition A function $f : A \longrightarrow B$ is said to be *surjective* (or onto) if each element of the codomain of the function is the image $f(x)$ of at least one element x of the domain A.

Example Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = 2x + 5$ and $g(x) = (x - 5)/2$. Now

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f((x-5)/2) \\ &= 2((x-5)/2) + 5 \\ &= x \end{aligned}$$

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(2x+5) \\ &= (2x+5-5)/2 \\ &= x \end{aligned}$$

Clearly $f \circ g = I$ and $g \circ f = I$ where I is the identity function on $\mathbb{R}$. Hence the function $f(x)$ has an inverse given by $g(x)$. Furthermore, the function $g(x)$ has an inverse given by $f(x)$. Draw the graph of each function on the same set of axes to notice the symmetry in the line $y = x$, i.e., the identity function $I_{\mathbb{R}}$.

Example Let $f : \mathbb{R}^+ \setminus \{0\} \longrightarrow \mathbb{R}$ defined by $f(x) = \log_{10} x$ and the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = 10^x$ Now

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(10^x) \\ &= \log_{10} 10^x \\ &= x \log_{10} 10 \\ &= x \end{aligned}$$

Clearly $f \circ g = I$ where I is the identity function on $\mathbb{R}$. Hence the function $f(x)$ has an inverse given by $g(x)$.

Example Let $f : \mathbb{R}^+ \setminus \{0\} \longrightarrow \mathbb{R}$ defined by $f(x) = \log_{\frac{1}{2}} x$ and the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = (\frac{1}{2})^x$. Now

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f((\frac{1}{2})^x) \\ &= \log_{\frac{1}{2}} (\frac{1}{2})^x \\ &= x \log_{\frac{1}{2}} \frac{1}{2} \\ &= x \end{aligned}$$

Clearly $f \circ g = I$ where I is the identity function on $\mathbb{R}$. Hence the function $f(x)$ has an inverse given by $g(x)$. Draw the graph of each function on the same set of axes to notice the symmetry in the line $y = x$, i.e., the identity function $I_{\mathbb{R}}$.

Exercise Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined by $f(x) = \sqrt{x}$ and the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = x^2$. Plot both graphs on the same pair of axes. Show that they are inverses of each other.

Remark Our functions so far have been single variable real valued functions

$$f : \mathbb{R} \longrightarrow \mathbb{R} : x \longrightarrow f(x)$$

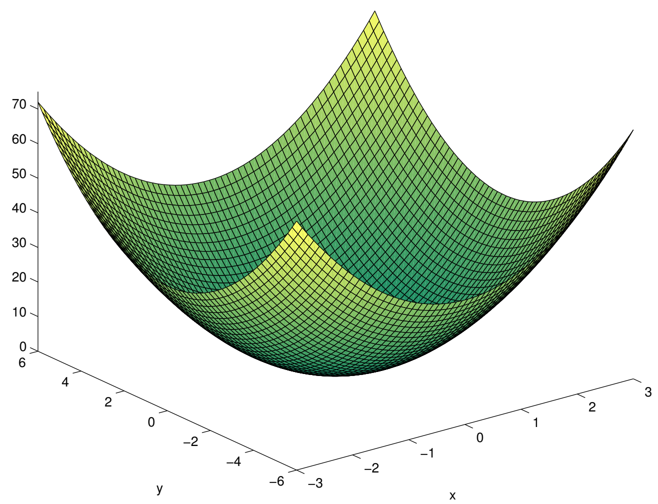
A function of two variables has the form

$$f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} : (x, y) \longrightarrow f(x, y)$$

Example The graph of this function

$$f(x, y) = 4x^2 + y^2$$

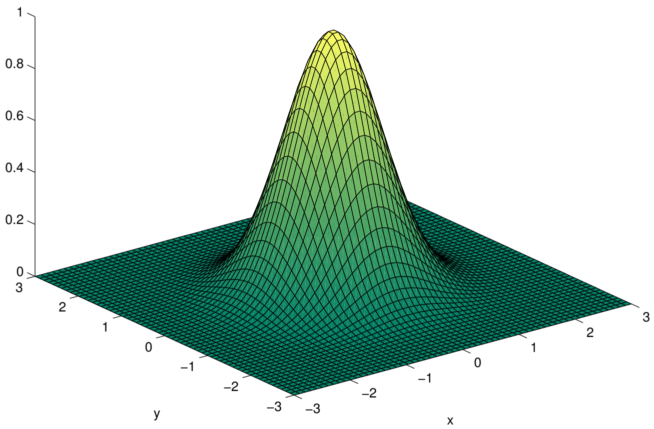
plotted in the interval $-3 \leq x \leq 3$ and $-6 \leq y \leq 6$ is as follows:



Example The graph of this function

$$f(x,y) = e^{-x^2-y^2}$$

plotted in the interval $-3 \le x \le 3$ and $-3 \le y \le 3$ is as follows:



Example The graph of this function

$$f(x,y) = y^2 - x^2$$

plotted in the interval $-5 \le x \le 5$ and $-10 \le y \le 10$ is as follows: