INSTITIÚID TEICNEOLAÍOCHTA CHEATHARLACH

INSTITUTE OF TECHNOLOGY CARLOW

QUATERNIONS AND ROTATIONS

# 1 Quaternions and Rotations

## 1.1 Introduction

William Rowan Hamilton (1805-1865) was considered by some of his contemporaries to approach Issac Newton in intellectual power. He was born on August $4^{th}$, 1805 in Dublin and possibly because of his father's financial circumstances had become strained, William was sent at the age of three to live with his uncle, the Reverend James Hamilton, head of a diocesan school in the village of Trim, Co. Meath. His uncle was a classical scolar who had graduated from Trinity College, Dublin. William had little contact with his parents during childhood. He began his education as soon as he arrived in Trim, quickly revealing himself to be a child prodigy. His uncle was aiming to get him into Trinity College in due course, and gave him every encouragement. Languages came easily to him and by the age of ten he was said to know not only the classical languages of Latin, Greek and Hebrew but also some modern European languages. His serious mathematical education began at the age of thirteen with the study of Euclid, and went on to read the works of Newton and Laplace.

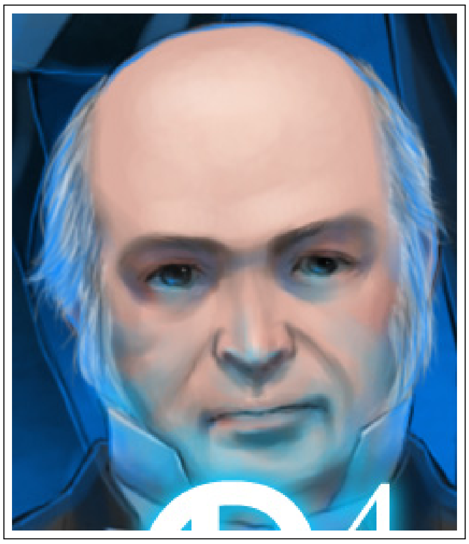


**Figure 1:** *William Rowan Hamilton (1805–1865)*

In 1824, at nineteen, Hamilton entered Trinity College and that same year published a note correcting a minor error in Laplace's work. During his four years at Trinity College he was awarded two separate 'optimes' for his performances: so rare was the honour that no optimes at all had been awarded for twenty years. In 1827 the chair of astronomy at Trinity became vacant and Hamilton was invited to apply. In spite of his age and lack of experience Hamilton was appointed Andrews professor of astronomy, and consequently became Astronomer Royal of Ireland and Director of the Observatory at Dunsink, although he had yet to take his degree. Hamilton retained these latter positions throughout his life, and made the observatory his home. After a few years he gave up his work as a practical astronomer to concentrate on mathematics. During the 1830s he led an extremely active life. These were his most productive years. He made important contributions to the understanding of dynamics and optics, invented quaternions and, in graph theory, developed what he called 'icosian calculus'.

It was the connection between the algebra of complex numbers and rotations in the plane that intrigued Hamilton for years. His problem was to try to discover an extension to the complex number system that could be used to describe rotations in 3–dimensions, using triples of real numbers instead of pairs. This he was unable to accomplish, for what later tuned out to be very good reason. On 16th October, 1843, Hamilton while walking with his wife along the Royal canal in Dublin, that he had a flash of inspiration when crossing Broome Bridge. In that instant he saw that triples were not enough, but that four-tuples were required. He saw that he needed not just the complex number component $i$, but rather three such components $i$, $j$ and $k$ satisfying the relationship


**Figure 2:** *Plaque (Broome Bridge), Royal Canal, Dublin.*

$$i^2 = j^2 = k^2 = ijk = -1$$

So struck was he by this discovery that supposedly he stopped and carved this equation into the stone of Broome Bridge. It was at that moment quaternions were discovered. The plaque to commemorate his discovery was erected on Broom Bridge and has the inscription

> *Here as he walked by on* $16^{th}$ *October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication*
>
> $$i^2 = j^2 = k^2 = ijk = -1$$
>
> & *cut it on a stone of this bridge.*

From 1843 the concept of a quaternion was simply another abstract mathematical idea until the advent of computer graphics almost 100 years later. They are now used in most computer games and graphics programs where objects pass through multiple rotations as part of the animation of the objects on the screen. This topic will attempt to show how complex numbers can be used to rotate in 2–dimensions and the extension of complex numbers to quaternions will allow for rotations in 3–dimensions. In doing so we develop the algebra of both complex numbers and quaternions. In defining a complex number we will introduce a new number $i$ such that $i^2 = -1$, i.e. $i = \sqrt{-1}$. This will give rise to further numbers of the form $x + iy$.

## 1.2 Complex Numbers

There is no real number $x$ such that $x^2 + 1 = 0$. The introduction of a new number $i$ such that $i^2 = -1$ gives rise to further numbers of the form $x + iy$. A number of the form $x + iy$, where x and y are real, is a *complex number*. The set of complex numbers is denoted by $\mathbb{C}$.

$$\mathbb{C} = \{z : z = x + iy : x, y \in \mathbb{R}, i^2 = -1\}$$

The real numbers $x$ and $y$ are referred to as the *real* and *imaginary* parts of the complex number $x + iy$. One adds and subtracts complex numbers by adding or subtracting their real parts, and adding or subtracting their imaginary parts. Thus

$$(x + iy) + (u + iv) = (x + u) + i(y + v)$$
$$(x + iy) - (u + iv) = (x - u) + i(y - v)$$

Multiplication of a complex number is defined such that

$$(x + iy) \times (u + iv) = (xu - yv) + i(xv + uy)$$

Division of a complex number is defined such that

$$\frac{(x + iy)}{(u + iv)} = \frac{(x + iy)(u - iv)}{(u + iv)(u - iv)} = \frac{xu + yv}{u^2 + v^2} + i\frac{yu - xv}{u^2 + v^2}$$

Hence the reciprocal $(x + iy)^{-1}$ of a non-zero complex number $x + iy$ is given by the formula

$$(x + iy)^{-1} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

Complex numbers may be represented by points of the plane (through he Argand diagram). A complex number $x + iy$ represents, and is represented by, the point of the plane whose Cartesian coordinated are $(x, y)$. One often therefore refers to the set of all complex numbers as the *complex plane*. This complex plane is pictured as a flat plane, containing lines, circles, etc., and distances and angles are defined in accordance with the usual principles of plane geometry and trigonometry. The *modulus* of a complex number $x + iy$ is defined to be the quantity $\sqrt{x^2 + y^2}$. It represents the distance of the corresponding point $(x, y)$ of the complex plane from the origin $(0, 0)$. The modulus of a complex number $z$ is denoted by $|z|$.

The *conjugate* of a complex number $z = x + iy$ is defined to be $\bar{z} = x - iy$. Notice that a complex number multiplied by its conjugate results in a real number.

$$z \times \bar{z} = \bar{z} \times z = x^2 + y^2$$

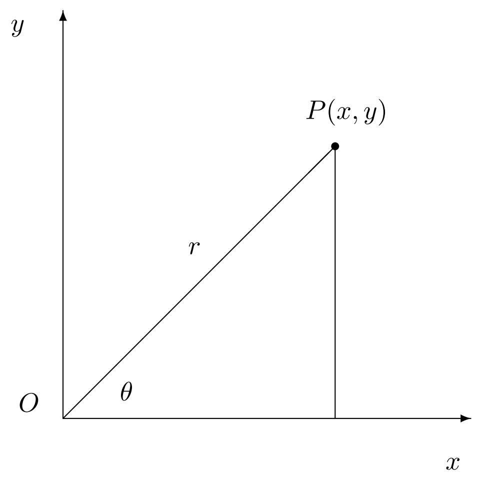
Finally, the *inverse* of $z$ is denoted as $z^{-1}$ and is defined as

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

Hence $z \times z^{-1} = 1$.

## 1.3 Polar Form of a Complex Number

In order to make a connection with quaternions, we consider the geometric interpretation of complex numbers. Any point in the plane may be specified by Cartesian coordinates $(x, y)$. The corresponding polar coordinates allow the point to be defined in terms of a magnitude and direction. This coordinate $(x, y)$ corresponds to the complex number $z = x + iy$.




Here the point $P$ has Cartesian coordinates $(x, y)$ and polar coordinates $(r, \theta)$; $r$, the modulus, is the distance of the point $P$ from the origin $O$, and $\theta$ is the direction angle which the line $OP$ makes with the positive x-axis, the anti-clockwise direction being reckoned positive. From the diagram, we see that

$$x = r\cos\theta$$

$$y = r\sin\theta$$

these equations express the Cartesian coordinates in terms of polar coordinates. The equations which express the polar coordinates in terms of the Cartesian coordinates are

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \cos^{-1}\left(\frac{x}{\sqrt{x^2 + y^2}}\right)$$

$$\theta = \sin^{-1}\left(\frac{y}{\sqrt{x^2 + y^2}}\right)$$

Note that there is only one value of $\theta$ in the interval $[0, 2\pi]$ such that $x = r\cos\theta$ and $y = r\cos\theta$; but there are infinitely many other values which also satisfy these equations, since $\cos\theta = \cos(\theta + 2n\pi)$ and $\sin\theta = \sin(\theta + 2n\pi)$ for some integer $n$.

The complex number $z = x + iy$ may be identified with the point $(x, y)$ in the plane, and so may be written in *polar form* as

$$z = r(\cos\theta + i\sin\theta)$$

In this polar form, the positive real number $r$ is simply the modulus of the complex number $z$, and is uniquely determined by $z$

$$r = \sqrt{x^2 + y^2} = |z|$$

On the other hand, $\theta$ is not uniquely determined by $z$: if $\theta_0$ is any real number such that

$$z = r(\cos(\theta_0 + 2n\pi) + i\sin(\theta_0 + 2n\pi))$$

for each integer $n$. Any one of these numbers $\theta_0 + 2n\pi$ is called an *argument* of $z$, but only one of them lies in the interval $[0, 2\pi]$; this unique number is called *the principle value of the argument of z*, and the corresponding polar form is sometimes called the *polar form of z.*

One consequence of the non-uniqueness of the argument of a complex number has to do with the criterion for equality between two complex numbers in polar form: we have that

$$r(\cos\theta + i\sin\theta) = s(\cos\phi + i\sin\phi) \Leftrightarrow \left\{ \begin{array}{l} r = s \;\; and \\ \theta = \phi + 2n\pi \;\;\; for\ some\ n \in \mathbb{Z}. \end{array} \right\}$$

**Exercise** Write each of the following complex numbers in *polar form*

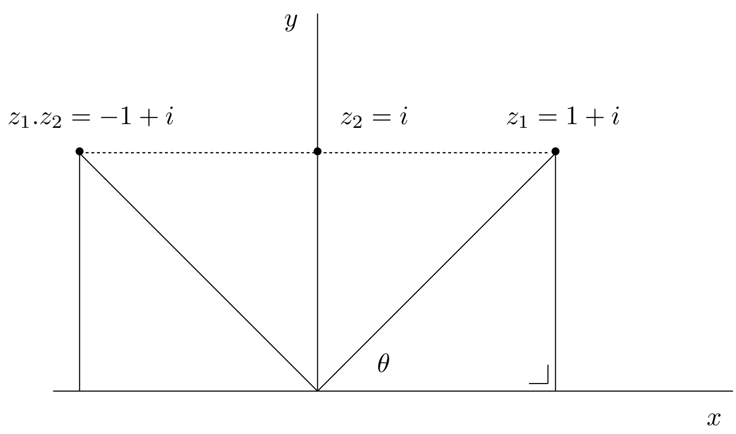
i $z = 3 + 3i$

ii $z = 2 + 2\sqrt{3}i$

iii $z = -\sqrt{6} - \sqrt{2}i$

**Remark** Often the geometric interpretation of a complex number $z = x + iy$ is made by identifying the complex number with a 2–dimensional vector $\vec{v} = (x, y)$, which we may think of as a vector from the origin to the point $(x, y)$ in the plane. In that case the magnitude of the complex number is simply the length of the vector and the angle of the complex number is the angle between the vector and the positive x axis. It is now easy to see that the addition of complex numbers corresponds exactly to the parallelogram law for vector addition, i.e., the rule for adding vectors is exactly the same as that for adding complex numbers. In the case of multiplication of complex numbers, the magnitude of the product is equal to the product of the moduli, and the angle of the product is the sum of the angles (arguments). If we limit ourselves to complex numbers of magnitude 1, then the multiplication of the complex numbers amounts to a rotation in the plane.

For example, let $z_1 = 1 + i$ and $z_2 = i$.




Notice that the magnitude of both $z_1$ and $z_2$ is 1. The arguments of $z_1$ and $z_2$ are $\pi/4$ and $\pi/2$ respectively. Multiplying the complex numbers will yield the complex number $z_1.z_2 = -1 + i$, which is the original complex number $z_1 = 1 + i$ rotated through $\pi/2$.

## 1.4 Quaternions

**Definition** A *quaternion* may be defined as an expression of the form

$$w + xi + yj + zk$$

where $w, x, y$ and $z$ are real numbers.

Thus a quaternion is the sum of a scalar and a vector in $\mathbb{R}^3$.

## 1.5 Quaternion Algebra

There are operations of addition, subtraction and multiplication defined on the set $\mathbb{H}$ of quaternions. These are binary operations on the set.

The definitions of addition and subtraction are straightforward.

The *sum* and *difference* of two quaternions $q = w + xi + yj + zk$ and $q' = w' + x'i + y'j + z'k$ are given by the formulae

$$\begin{aligned} q + q' &= (w + xi + yj + zk) + (w' + x'i + y'j + z'k) \\ &= (w + w') + (x + x')i + (y + y')j + (z + z')k \end{aligned}$$

$$\begin{aligned} q - q' &= (w + xi + yj + zk) - (w' + x'i + y'j + z'k) \\ &= (w - w') + (x - x')i + (y - y')j + (z - z')k \end{aligned}$$

These operations of addition and subtraction of quaternions are binary operations on the set $\mathbb{H}$ of quaternions. It is easy to see that the operation of addition is commutative and associative, and that the *zero quaternions* $0 + 0i + 0j + 0k$ is an identity element for the operation of addition. The operation of subtraction of quaternions is neither commutative nor associative. This results directly from the fact that the operation of subtraction on the set of real numbers is neither commutative nor associative. Let $q$ be a quaternion. Then $q = w + xi + yj + zk$ for some real numbers $w, x, y$ and $z$, then there is a corresponding quaternion $-q$, with $-q = (-w) + (-x)i + (-y)j + (-z)k$. Then $q + (-q) = (-q) + q = 0$, where 0 here denotes the zero quaternion $0 + 0i + 0j + 0k$. Thus, to every quaternion $q$ there corresponds a quaternion $-q$ that is the inverse of $q$ with respect to the operation of addition.

Let $q$ be a quaternion and $\lambda$ a scalar. Then

$$\begin{aligned} \lambda q &= \lambda(w + xi + yj + zk) \\ &= \lambda w + \lambda xi + \lambda yj + \lambda zk \end{aligned}$$

**Example** Given the quaternions

$$\begin{aligned} q_1 &= 2 + i + 3j + 0k \\ q_2 &= -1 + 4i - 2j + k \end{aligned}$$

So, for example

$$\begin{aligned} q_1 + q_2 &= (2 + i + 3j + 0k) + (-1 + 4i - 2j + k) \\ &= (2 - 1) + (1 + 4)i + (3 - 2)j + (0 + 1)k \\ &= 1 + 5i + j + k \end{aligned}$$

$$\begin{aligned} 2q_1 - 4q_2 &= 2(2 + i + 3j + 0k) - 4(-1 + 4i - 2j + k) \\ &= (4 + 2i + 6j + 0k) - (-4 + 16i - 8j + 4k) \\ &= (4 + 4) + (2 - 16)i + (6 + 8)j + (0 - 4)k \\ &= 8 - 14i + 14j - 4k \end{aligned}$$

The definition of quaternion multiplication is somewhat more complicated than the definition of addition and subtraction. The *product* of two quaternions $q = w+xi+yj+zk$ and $q' = w'+x'i+y'j+z'k$ are given by the formulae

$$\begin{aligned} & (w + xi + yj + zk) \times (w' + x'i + y'j + z'k) \\ = \; & (ww' - xx' - yy' - zz') + (wx' + xw' + yz' - zy')i \\ & +(wy' + yw' + zx' - xz')j + (wz' + zw' + xy' - yx')k \end{aligned}$$

We shall often denote the product $q \times q'$ of quaternions $q$ and $q'$ by $qq'$.

Given any real number $w$, let us denote the quaternion $w + 0i + 0j + 0k$ by $w$ itself. Let us also denote the quaternions $0 + 1i + 0j + 0k$, $0 + 0i + 1j + 0k$, $0 + 0i + 0j + 1k$ by $i, j$ and $k$ respectively. It follows directly from the above formula defining multiplication of quaternions that

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k \quad , \quad jk = -kj = i \quad , \quad ki = -ik = j$$

It follows directly from these identities that

$$ijk = -1$$

The operation of multiplication on the set $\mathbb{H}$ of quaternions is **not** commutative.

**Example** Given the quaternions

$$\begin{aligned} q_1 &= 2 + 3i + 0j + k \\ q_2 &= 4 + 2i + 3j + 0k \end{aligned}$$

Now

$$\begin{aligned} q_1 \times q_2 &= (2 + 3i + 0j + k) \times (4 + 2i + 3j + 0k) \\ &= (8 + 4i + 6j) + (12i + 6i^2 + 9ij) + (4k + 2ki + 3kj) \\ &= (8 - 6) + 13i + 8j + 13k \\ &= 2 + 13i + 8j + 13k \end{aligned}$$

**Example** Given the quaternions

$$\begin{aligned} q_1 &= 1 - 2i + j + 3k \\ q_2 &= 0 + 2i - j + 4k \end{aligned}$$

Now

$$\begin{aligned} q_1 \times q_2 &= (1 - 2i + j + 3k) \times (0 + 2i - j + 4k) \\ &= (2i - j + 4k) + (-4i^2 + 2ij - 8ik) + (2ij - j^2 + 4jk) + (6ki - 3kj + 12k^2) \\ &= (4 + 1 - 12) + (2 + 4 + 3)i + (-1 + 8 + 6)j + (4 + 2 - 2)k \\ &= -7 + 9i + 13j + 4k \end{aligned}$$

**Remark** The quaternion $1 + 0i + 0j + 0k$ is an identity element for the operation of multiplication.

Let $q$ be a quaternion. Then $q = w + xi + yj + zk$ for some real numbers $w, x, y$ and $z$. We define the *conjugate* $\bar{q}$ of $q$ to be the quaternion $\bar{q} = w - xi - yj - zk$. The definition of quaternion multiplication may then be used to show that $q \times \bar{q} = \bar{q} \times q = w^2 + x^2 + y^2 + z^2$.

We define the *modulus* $|q|$ of the quaternion $q$ by the formula

$$|q| = \sqrt{w^2 + x^2 + y^2 + z^2}$$

Then $q \times \bar{q} = \bar{q} \times q = |q|^2$ for all quaternions q. If $q$ is a non-zero quaternion, and if the quaternion $q^{-1}$ is defined by the formula

$$q^{-1} = \frac{1}{|q|^2}\bar{q}$$

then $q \times q^{-1} = q^{-1} \times q = 1$. We conclude therefore that, given any non-zero quaternion $q$, there exists a quaternion $q^{-1}$ that is the inverse of $q$ with respect to multiplication.

**Example** Given the quaternion

$$q = 2 - 3i + j + 4k$$

we find that the congugate

$$\bar{q} = 2 + 3i - j - 4k$$

**Example** Given the quaternion

$$q = 2 - 3i + j + 4k$$

we can determine the inverse of $q$ and show that $q \times q^{-1} = 1 + 0i + 0j + 0k$ , the identity element. Firstly to determine the inverse of $q$ we note that $|q|^2 = 30$ and $\bar{q} = 2 + 3i - j - 4k$, hence

$$q^{-1} = \frac{1}{|q|^2}\bar{q} = \frac{1}{30}(2 + 3i - j - 4k)$$

To check our work, note that

$$q \times q^{-1} = \frac{1}{30}(30 + 0i + 0j + 0k) = 1 + 0i + 0j + 0k$$

as required.

**Exercise** Given the quaternions

$$q_1 = 1 + i - 4j + 6k$$
$$q_2 = 2 + 4i - 0j + 3k$$

Evaluate

$$q_1 \times q_2 \times \bar{q}_1$$

where $\bar{q}$ denotes the conjugate of the quaternion $q$. Show that $q_1 \times q_1^{-1} = 1$.

## 1.6 Quaternions and Rotations

Let $r(x, y, z)$ be the Cartesian coordinates of a point in three-dimensional space. This point may be represented by a quaternion with the real part equal to zero, i.e.,

$$r = 0 + xi + yj + zk$$

If this is now multiplied by another quaternion $q$, then the product $q.r$ will have a non-zero real part. However, the product $q.r.q^{-1}$ will return to having a real part equal to zero. This is easily seen since the real part of $q.r$ is always equal to the real part of $r.q$, so therefore

$$Real(q.r.q^{-1}) = Real(r.q.q^{-1}) = Real(r.1) = 0$$

Thus we can conclude that any rotation of a point in three-dimensional space using a quaternion $q$ will be obtained by a multiplication of the form $q.r.q^{-1}$. Without going into detail here, it can be shown that the following quaternion q

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(li + mj + nk)$$

will rotate a point in three-dimensional space through $\theta$ degrees, *anti-clockwise* about any axis $\vec{n} = (l, m, n)$, where $\vec{n}$ is a unit vector. Let $\bar{q}$ be the conjugate of $q$, given by the formula

$$\bar{q} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}(li + mj + nk)$$

Let $r(x, y, z)$ and $r'(x', y', z')$ be the Cartesian coordinates of two points in three-dimensional space. Representing these points as quaternions we have

$$\begin{aligned} r &= xi + yj + zk \\ r' &= x'i + y'j + z'k \end{aligned}$$

We can show that if $r' = q.r.q^{-1}$ then a rotation about the axis $\vec{n} = (l, m, n)$ through an angle $\theta$ will send the point $r(x, y, z)$ to the point $r'(x', y', z')$. (The effect of a rotation through an angle $\theta$ in the opposite sense can be achieved by replacing $\theta$ by $-\theta$ in the definition of the quaternion $q$). In this way the algebra of quaternions may be used in areas of application such as computer-aided design and the programming of computer games, in order to calculate the results of rotations applied to points in three-dimensional space.

## 1.7 Some Examples

**Example** Using a quaternion, we can rotate the point $r(1,2,3)$ in three-dimensional space through $120^o$ *anti-clockwise* about the axis $\vec{v} = (1,1,1)$ as follows.

Firstly,

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(li + mj + nk)$$

In this example

$$\vec{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Hence

$$q = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k\right)$$

simplifying, we get

$$q = \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k$$

Now

$$\bar{q} = \frac{1}{2} - \frac{1}{2}i - \frac{1}{2}j - \frac{1}{2}k$$

Recall

$$q^{-1} = \frac{1}{|q|^2}\bar{q}$$

Also

$$|q|^2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

Therefore

$$q^{-1} = \frac{1}{2} - \frac{1}{2}i - \frac{1}{2}j - \frac{1}{2}k$$

Finally the rotation is obtained from the quaternion multiplication $q.r.q^{-1}$. Representing the point $r(1,2,3)$ as a quaternion we have

$$r = 0 + i + 2j + 3k$$

Finally

$$\begin{aligned} q.r.q^{-1} &= \left(\frac{1}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k\right).\left(i+2j+3k\right).\left(\frac{1}{2}-\frac{1}{2}i-\frac{1}{2}j-\frac{1}{2}k\right) \\ &= \left(-3+i+0j+2k\right).\left(\frac{1}{2}-\frac{1}{2}i-\frac{1}{2}j-\frac{1}{2}k\right) \\ &= \left(0+3i+1j+2k\right) \end{aligned}$$

So the three-dimensional point $(1,2,3)$ gets rotated to $(3,1,2)$.

**Example** Using a quaternion we can rotate the point $r(1,2,3)$ through $90^o$ *anti-clockwise* about the z-axis i.e., $\vec{v}=(0,0,1)$. Determine the components of this point after this rotation.

Firstly,

$$q=\cos\frac{\theta}{2}+\sin\frac{\theta}{2}(li+mj+nk)$$

In this example

$$\vec{n}=(0,0,1)$$

Hence

$$q=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}k$$

Now

$$\bar{q}=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}k$$

Recall

$$q^{-1}=\frac{1}{|q|^2}\bar{q}$$

Also

$$|q|^2=\frac{1}{2}+\frac{1}{2}=1$$

Therefore

$$q^{-1}=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}k$$

Finally the rotation is obtained from the quaternion multiplication $q.r.q^{-1}$. Representing the point $r(1,2,3)$ as a quaternion we have

$$r=0+i+2j+3k$$

Finally

$$\begin{aligned} q.r.q^{-1} &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}k\right).\left(i + 2j + 3k\right).\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}k\right) \\ &= \left(\frac{-3}{\sqrt{2}} - \frac{1}{\sqrt{2}}i + \frac{3}{\sqrt{2}}j + \frac{3}{\sqrt{2}}k\right).\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}k\right) \\ &= \left(0 - 2i + 1j + 3k\right) \end{aligned}$$

So the three-dimensional point $(1, 2, 3)$ gets rotated to $(-2, 1, 3)$.

**Remark** A quaternion is another mathematical structure that is used by graphics programmers to represent 3-dimensional rotations. It has the following advantages over matrices:

1. Quaternions require less storage space – a quaternion only requires 4 variables or a single array with 4 elements, whereas the corresponding $3 \times 3$ rotational matrix requires 9 elements in an array and so takes up more memory.

2. To carry out two or more sequential rotations requires less arithmetic operations with quaternions than with matrix multiplication. To combine two rotations together using quaternions, simply multiply $q_1.q_2$ and this takes 28 multiplication/addition operations. However to multiply two rotational matrices $R_1.R_2$ takes 45 multiplication/addition operations. So, if a graphics program requires multiple rotations, the execution time is almost halved using quaternions.

3. Possibly the main advantage in the use of quaternions is found when rotational animation of an object on the screen is required. Quaternions is the only format that will allow for a smooth transition (called interpolation) to occur as an object rotates from one position to another.

4. One further major advantage is that using quaternions it is possible to avoid the problem of **gimble lock**. Gimbal lock occurs when 2 or more rotations are used to achieve a new orientation for an object. It can happen that two of the three $X$, $Y$ and $Z$ axes can end up pointing in the same direction.

There are also some disadvantages to using quaternions.

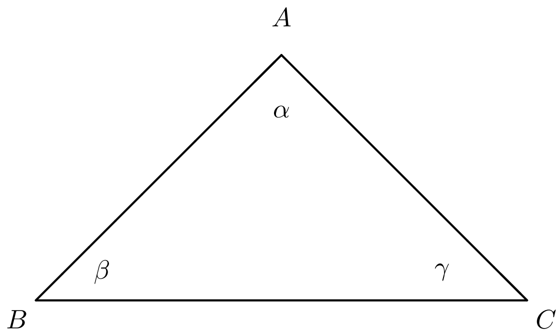
1. The main disadvantage in using quaternions is that it is very difficult for humans to visualise what exactly is going on.

2. Occasionally with a lot of quaternion calculations it is possible for round-off error to cause invalid values (e.g., division by zero errors). This is normally avoided by ensuring that all quaternions are normalised at each step in the calculation, i.e., the quaternions are kept to a unit length.

**Exercise** Using *quaternions* rotate the triangle ABC whose vertices are the points

$$A(1,-1,0) \quad , \quad B(2,1,-1) \quad , \quad C(-1,1,2)$$

through $60^o$ *anti-clockwise* about the z-axis.




Determine the area of the triangle ABC. Confirm that the area of this triangle remains unchanged as a result of the rotation.

[**Solution**: $A^{'}(1\cdot 366, 0\cdot 366, 0) \quad , \quad B^{'}(0\cdot 134, 2\cdot 23, -1) \quad , \quad C^{'}(-1\cdot 366, -0\cdot 366, 2)$]

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