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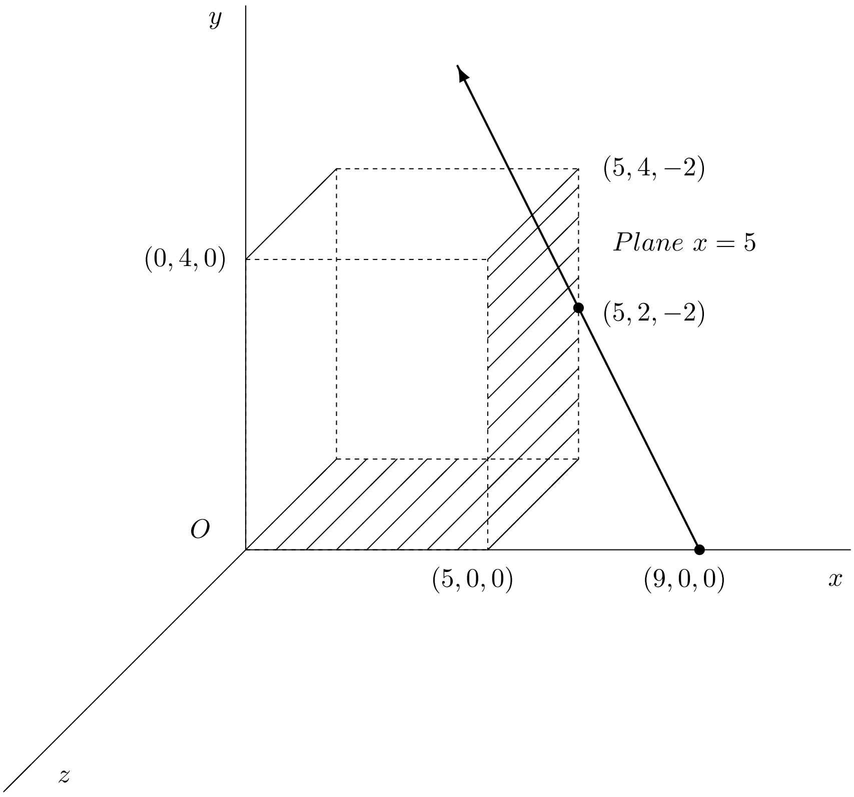
RAY TRACING

Contents



1 Ray Tracing

The term ray tracing means any algorithm that calculates where a beam of light interacts with objects and how the light is affected. Under this heading we look at how the point of intersection of a ray of light with objects can be determined and how the ray is altered by reflection and refraction at the surface of the object.

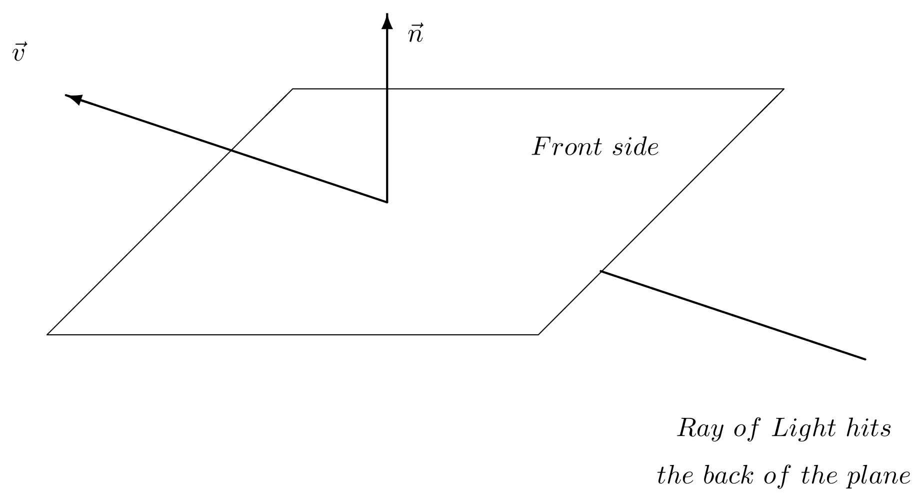
We define a ray, using the starting point $\vec{P_0}$ and in the direction $\vec{v}$, to be the set of all point $P(x, y, z)$ for which

$$\begin{aligned} \overrightarrow{P_0P} &= t.\vec{v} \\ \vec{P} - \vec{P_0} &= t.\vec{v} \\ \vec{P} &= \vec{P_0} + t.\vec{v} \qquad , \ t \in \mathbb{R}. \end{aligned}$$

We now attempt to determine to calculate the point of intersection of a ray with various objects in the objects coordinate system. It can then be translated back to the real world or camera coordinates as required.

1.1 Intersection of a ray and a Plane

In determining the point of intersection of a line (ray of light) and a plane in $\mathbb{R}^3$, it is important to determine which side of the plane that the line is coming from, so we define the **front** of the plane to be the side from which the normal vector $\vec{n}$ is pointing.



If the ray hits the back of the plane, then the angle between $\vec{n}$ and $\vec{v}$ (the direction of the ray) will be less than 90^o and so $\vec{n}.\vec{v}$ is **positive**. On the other hand, if the ray hits the front of the plane, then the angle between $\vec{n}$ and $\vec{v}$ (the direction of the ray) will be greater than 90^o and so $\vec{n}.\vec{v}$ is **negative**.

Example A ray Ω is defined by the parametric equations

$$\begin{aligned} x &= 1+t \\ y &= 0 \\ z &= -2t \end{aligned}$$

Note that we can also represent as $\vec{P} = (1,0,0) + t.(1,0,-2)$ where $t \in \mathbb{R}$.

Let the plane Π be $x + 2y - z - 3 = 0$.

Now to calculate the point of intersection of the line Ω and the plane Π.

$$\begin{aligned} (1+t) + 2(0) - (-2t) - 3 &= 0 \\ 3t &= 2 \\ t &= \frac{2}{3} \end{aligned}$$

Hence, the point of intersection is

$$\begin{aligned} x &= 1 + \frac{2}{3} = \frac{5}{2} \\ y &= 0 \\ z &= -2\left(\frac{2}{3}\right) = \frac{-4}{3} \end{aligned}$$

Hence the point of intersection is

$$\left(\frac{5}{2}, 0, -\frac{4}{3}\right)$$

Now, a normal to the plane Π is $\vec{n} = (1,2,-1)$. Since

$$\begin{aligned} \vec{n}.\vec{v} &= (1,2,-1).(1,0,-2) \\ &= 3 \end{aligned}$$

this is positive, so the ray must hit the plane Π from behind.

Exercise Where does the ray defined as

$$\vec{P} = (5, 5, 5) + t.(1, 3, -2)$$

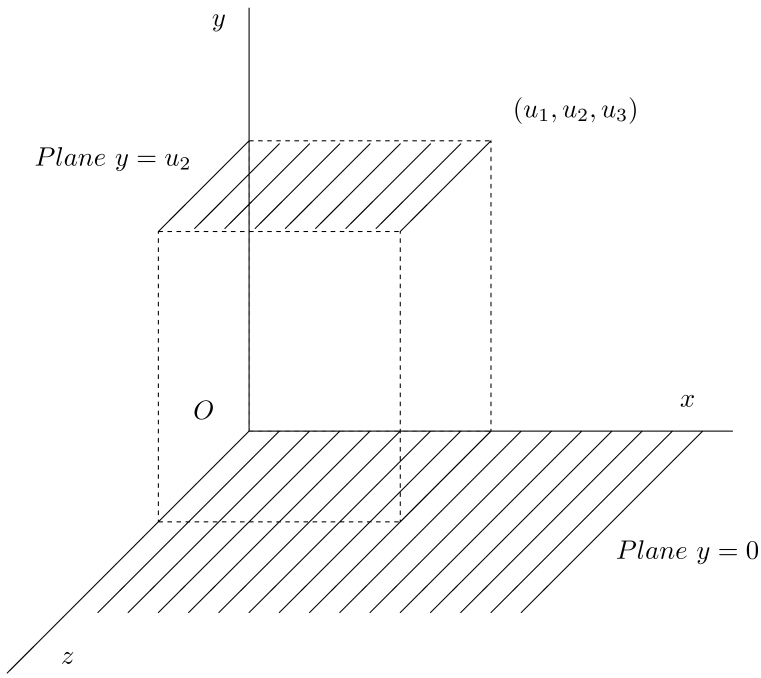
hit the plane $z = 0$. Does it hit from the front or behind the plane?

Exercise Where does the ray defined as

$$\vec{P} = (0, 0, 0) + t.(2, 1, 4)$$

hit the plane $x + y = 0$. Does it hit from the front or behind the plane?

1.2 Intersection of a ray and a Box



In the coordinate space of a box, putting the origin at one of its corners, its six sides will be described by the six planes

$$\begin{aligned} x &= 0 \quad , \quad x = u_1 \\ y &= 0 \quad , \quad y = u_2 \\ z &= 0 \quad , \quad z = u_3 \end{aligned}$$

To determine the intersection of a ray

$$\vec{P} = \vec{P_0} + t.\vec{v}$$

with this box, we only need consider the intersection of the ray with at most three of the sides, since the starting point, $\vec{P_0}$ and the direction $\vec{v}$ of the ray will determine which sides of the box are facing the ray and which are away from the ray. For example, if $\vec{v} = (v_1, v_2, v_3)$ and if both v_1 and p_1 are positive then the side $x = u_1$ is the only side parallel to the yz-plane that needs to be considered. Similarly the sides parallel to the xy-plane and the xz-planes can be reduced to one in each case.

After finding where the ray intersects with the plane, it must then be determined if the intersection point is on the side of the box or somewhere on the plane but outside the bow. This is best seen by example.

Example To determine if the ray

$$\vec{P} = (9,0,0) + t(-1, 0 \cdot 5, -0 \cdot 5)$$

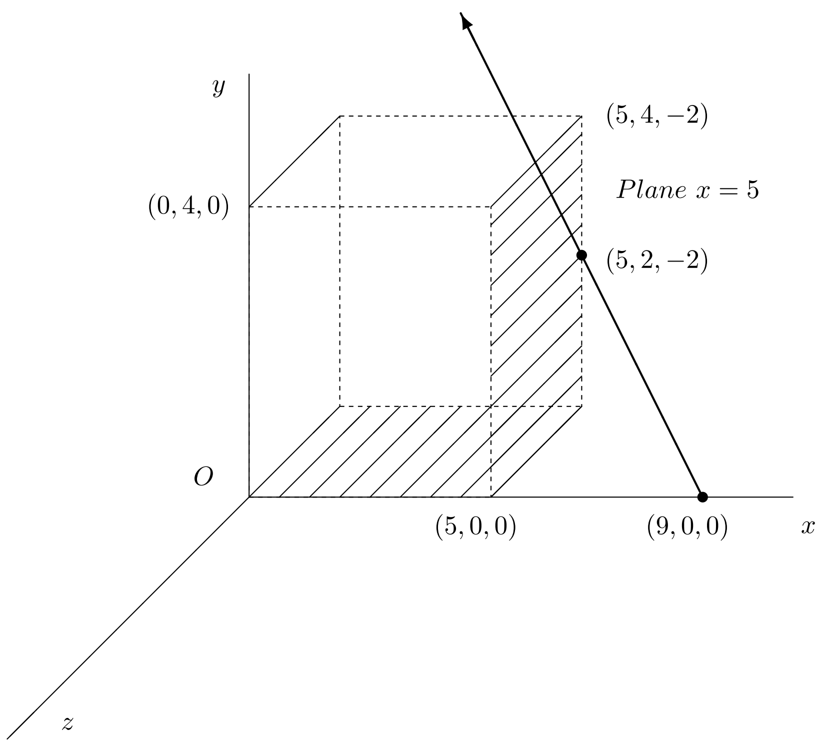
intersects with the box defined by the planes $x = 0$, $x = 5$, $y = 0$, $y = 4$, $z = 0$, $z = -2$ we note that the direction of the ray $\vec{v} = (-1, 0 \cdot 5, -0 \cdot 5)$ is changing in all three directions starting from the point $(9,0,0)$, so it could hit the outside of the box at $x = 5$ or $y = 4$ or $z = -2$.

To determine if it hits the box on the plane $x = 5$:

The parametric equations for the ray is $\vec{P}(t) = (9 - t, 0 \cdot 5t, -0 \cdot 5t)$. Now

$$\begin{aligned}
\vec{P}(1) &= (8, 0 \cdot 5, -0 \cdot 5) \textit{ which is outside the box} \\
\vec{P}(2) &= (7, 1, -1) \textit{ which is outside the box} \\
\vec{P}(3) &= (6, 1 \cdot 5, -1 \cdot 5) \textit{ which is outside the box} \\
\vec{P}(4) &= (5, 2, -2) \textit{ which just hits the edge of the box.}
\end{aligned}$$

Since the z-value of the direction $\vec{v} = -0 \cdot 5$, the ray cannot hit the box anymore, it is moving away behind the edge of the cube, as shown below.



Example To determine if the ray

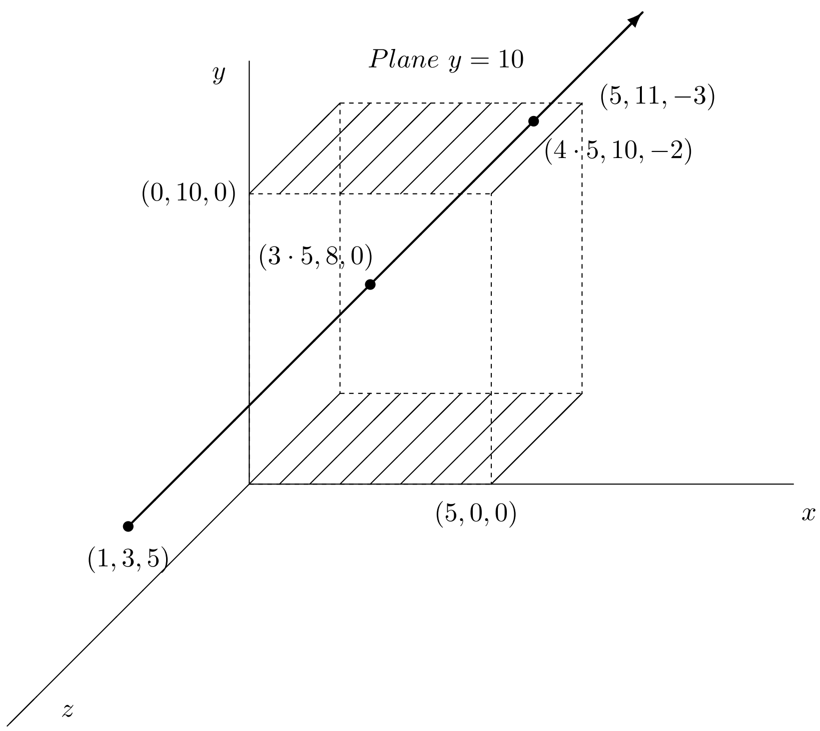
$$\vec{P} = (1,3,5) + t(0 \cdot 5, 1, -1)$$

intersects with the box defined by the planes $x = 0$, $x = 5$, $y = 0$, $y = 10$, $z = 0$, $z = -3$ we note that the direction of the ray $\vec{v} = (0 \cdot 5, 1, 1)$ is changing in all three directions starting from the point $(1, 3, 5)$, so it could hit the outside of the box at $x = 5$ or $y = 10$ or $z = -3$.

The parametric equations for the ray is $\vec{P}(t) = (1 + 0 \cdot 5t, 3 + t, 5 - t)$. Now

$$\begin{aligned}
\vec{P}(1) &= (1 \cdot 5, 4, 4) \textit{ which is outside the box} \\
\vec{P}(2) &= (2, 5, 3) \textit{ which is outside the box} \\
\vec{P}(3) &= (2 \cdot 5, 6, 2) \textit{ which is outside the box} \\
\vec{P}(4) &= (3, 7, 1) \textit{ which is outside the box} \\
\vec{P}(5) &= (3 \cdot 5, 8, 0) \textit{ which is the plane } z = 0 \\
\vec{P}(6) &= (4, 9, -1) \textit{ which is inside the box} \\
\vec{P}(7) &= (4 \cdot 5, 10, -2) \textit{ which is the top y plane } y = 10 \\
\vec{P}(8) &= (5, 11, -3) \textit{ which is the right x plane } x = 5 \textit{ outside box}
\end{aligned}$$

The direction of the ray is given as $\vec{v} = (0 \cdot 5, 1, 1)$



The ray firsts hits the **front z-plane**, $z = 0$, at $(3 \cdot 5, 8, 0)$, then it hits the **top y-plane**, $y = 10$, at $(4 \cdot 5, 10, -2)$. The ray hits the **right side** x-plane at $(5, 11, -3)$ outside the box.

Exercise Determine if the ray

$$\vec{P} = (8, 7, 6) + t(-1, -1, -1)$$

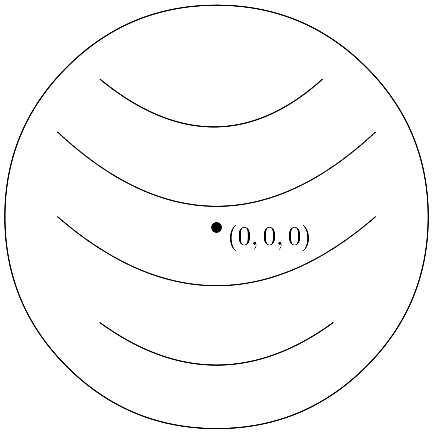
intersects with the box defined by the planes $x = 0$, $x = 5$, $y = 0$, $y = 10$, $z = 0$, $z = 3$ and if so, determine the points of intersection.

Solution: The ray hits the right **x-plane** at $(5, 4, 3)$, when $t = 3$, which is just on the front right edge of the box. It then hits the **z-plane** at $(2, 1, 0)$, when $t = 6$ and leaves the box at this point. It hits the base **y-plane** at $(1, 0, -1)$, when $t = 7$, which is outside the box.

1.3 Intersection of a ray and a Sphere

A *sphere* of radius r and centre at the origin $(0, 0, 0)$ is described by the equation

$$x^2 + y^2 + z^2 = r^2$$



Consider a ray

$$\vec{P} = \vec{P_0} + t.\vec{v}$$

i.e.,

$$\begin{aligned}\vec{P} &= (x_0, y_0, z_0) + t(v_1, v_2, v_3) \\ &= (x_0 + tv_1, y_0 + tv_2, z_0 + tv_3)\end{aligned}$$

This ray will intersect the sphere when

$$(x_0 + tv_1)^2 + (y_0 + tv_2)^2 + (z_0 + tv_3)^2 = r^2$$

where $\vec{P_0} = (x_0, y_0, z_0)$ is the start of the ray and $\vec{v} = (v_1, v_2, v_3)$ is the direction.

Multiplying out and rearranging we find that

$$\begin{aligned}(x_0 + tv_1)^2 + (y_0 + tv_2)^2 + (z_0 + tv_3)^2 &= r^2 \\ x_0^2 + 2tv_1x_0 + t^2v_1^2 + y_0^2 + 2tv_2y_0 + t^2v_2^2 + z_0^2 + 2tv_3z_0 + t^2v_3^2 &= r^2 \\ (v_1^2 + v_2^2 + v_3^2)t^2 + 2(v_1x_0 + v_2y_0 + v_3z_0)t + (x_0^2 + y_0^2 + z_0^2) &= r^2 \\ ||\vec{v}||^2t^2 + 2(\vec{P_0}.\vec{v})t + ||\vec{P_0}||^2 &= r^2\end{aligned}$$

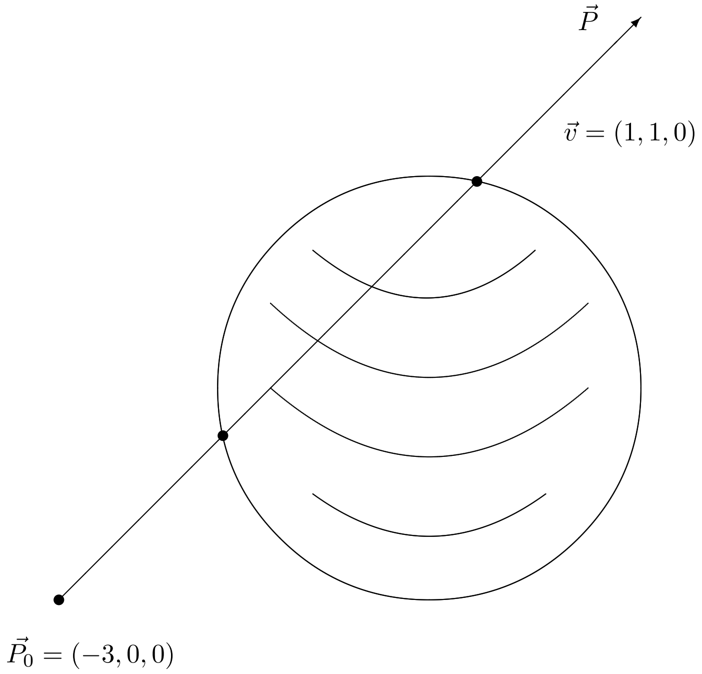
Hence, the intersection is at the value of t where

$$||\vec{v}||^2 t^2 + 2(\vec{P_0}.\vec{v})t + ||\vec{P_0}||^2 - r^2 = 0$$

and this equation we will use to find the point(s) of intersection of a line and a sphere. Notice that this equation is a *quadratic equation* and hence has three possibilities for a solution, i.e., no solution, one solution or two solutions.

Example To find where the ray intersects the sphere $\vec{P} = (-3, 0, 0) + t.(1, 1, 0)$ sphere

$$x^2 + y^2 + z^2 = 5$$



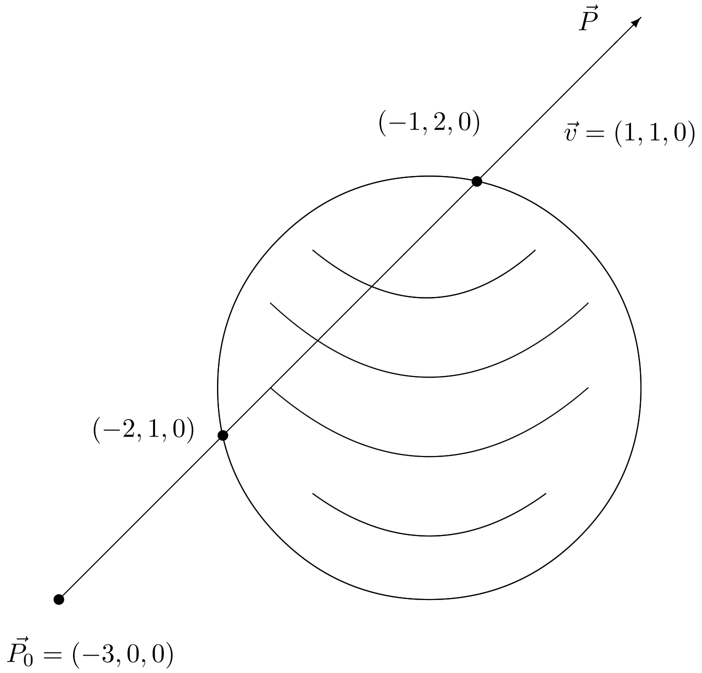
we determine the value of t where $||\vec{v}||^2 t^2 + 2(\vec{P_0}.\vec{v})t + ||\vec{P_0}||^2 - r^2 = 0$. Note that $r^2 = 5$ in this example. Also $||\vec{v}||^2 = 2$ and $||\vec{P_0}||^2 = 9$. Now

$$2t^2 + 2(-3)t + 9 - 5 = 0$$

Solving $2t^2 - 6t + 4 = 0$ yields $t = 1$ and $t = 2$. The parametric equations for the ray is $\vec{P} = (-3, 0, 0) + t.(1, 1, 0)$, hence the two points of intersection are

$$\begin{aligned} \vec{P}(1) &= (-3, 0, 0) + 1(1, 1, 0) = (-2, 1, 0) \\ \vec{P}(2) &= (-3, 0, 0) + 2(1, 1, 0) = (-1, 2, 0) \end{aligned}$$

Since the ray starts at the point $(-3, 0, 0)$, the first intersection is at the point $(-2, 1, 0)$ which is clearly the nearer of the two intersection points



Exercise

i Find where the ray $\vec{P} = (0, 0, -3) + t.(1, 1, 2)$ intersects the sphere $x^2 + y^2 + z^2 = 16 \cdot 5$

ii Find where the ray $\vec{P} = (1, 1, 0) + t.(4, 4, 4)$ intersects the sphere $x^2 + y^2 + z^2 = 1$.

iii Find where the ray $\vec{P} = (-3, 2, 1) + t.(1, 0, 0)$ intersects the sphere $x^2 + y^2 + z^2 = 5$.

Solutions:

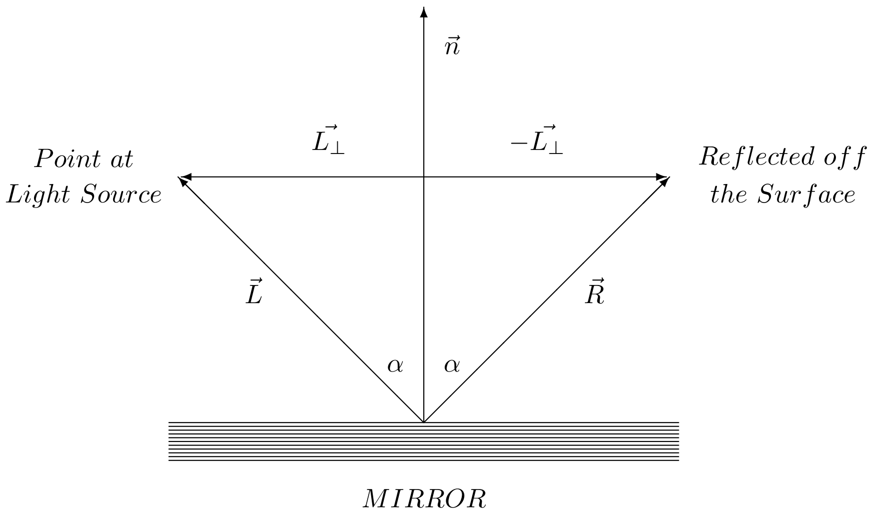
i The ray intersects with the sphere at $(-0 \cdot 5, -0 \cdot 5, -4)$ *and* $(2 \cdot 5, 2 \cdot 5, 2)$.

ii The ray does **not** *intersect the sphere.*

iii There is just one point of intersection at $(0, 2, 1)$.

1.4 Reflection of a ray from a Reflective Surface

It is often necessary to calculate the equation for a ray after it has been reflected from a mirror (flat surface) or some other shiny surface. Consider the following diagram



The vector $\vec{L}$ is a vector **pointing towards** the light source and the vector $\vec{R}$ represents the direction of the reflected light. If the normal to the surface is the vector $\vec{n}$, then **making all three all three vectors of unit length**, i.e., length equal 1, the component if $\vec{L}$ parallel to $\vec{n}$ is $\vec{L_\parallel} = (\vec{n}.\vec{L})\vec{n}$ and so the component perpendicular to $\vec{n}$ is

$$\begin{aligned}\vec{L_\perp} &= \vec{L} - \vec{L_\parallel} \\ &= \vec{L} - (\vec{n}.\vec{L})\vec{n}\end{aligned}$$

From the above diagram it is clear that the reflected vector $\vec{R} = \vec{L} - 2\vec{L_\perp}$, so the equation for the reflected vector can always be found using the following equation

$$\begin{aligned}\vec{R} &= \vec{L} - 2(\vec{L} - (\vec{n}.\vec{L})\vec{n}) \\ &= \vec{L} - 2\vec{L} + 2(\vec{n}.\vec{L})\vec{n} \\ &= 2(\vec{n}.\vec{L})\vec{n} - \vec{L}\end{aligned}$$

So, in its simplest form the reflected ray from a flat surface (e.g. a mirror) is represented as

$$\vec{R} = 2(\vec{n}.\vec{L})\vec{n} - \vec{L}$$

Example The surface of a mirror is defined by the plane $y = 4$ and the light is coming from the direction

$$\vec{v} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

To find the direction of the reflected ray first notice that the plane $y = 4$ has normal vector $\vec{n} = (0, 1, 0)$ which is a unit vector. The vector $\vec{L}$ is a vector **pointing towards** the light source, hence

$$\vec{L} = -\vec{v} = \left(\frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$$

Now

$$\begin{aligned}
\vec{R} &= 2(\vec{n}.\vec{L})\vec{n} - \vec{L} \\
&= 2\left(\frac{-1}{\sqrt{6}}\right)\vec{n} - \vec{L} \\
&= \left(\frac{-2}{\sqrt{6}}\right)(0, 1, 0) - \left(\frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right) \\
&= \left(0, \frac{-2}{\sqrt{6}}, 0\right) - \left(\frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right) \\
\vec{R} &= \left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)
\end{aligned}$$

Notice that comparing the incoming light direction

$$\vec{v} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

to the reflected direction

$$\vec{R} = \left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

only the y component has been reversed. This is to be expected for the reflection off a flat horizontal plane $y = 4$.

Exercise The surface of a mirror is defined by the plane $2x - z = -2$, and light is coming from the direction

$$\vec{v} = \left(\frac{1}{2}, \frac{-1}{\sqrt{2}}, \frac{1}{2}\right)$$

Find the direction of the reflected ray.

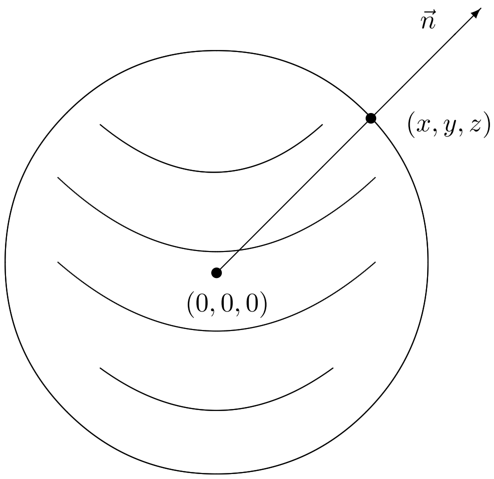
Solution: Recall to write $\vec{n}$ as a normal vector.

$$\vec{R} = \left(\frac{1}{10}, \frac{-1}{\sqrt{2}}, \frac{7}{10}\right)$$

Remark If any surface is circular in shape, then the normal vector $\vec{n}$ at the surface will be pointing directly **away from the centre of the sphere**. Hence, for any point (x, y, z) on the surface of the sphere, the normal vector will point in the direction of the position vector of the point on the surface, so

$$\vec{n} = (x, y, z)$$

Notice that this vector will not usually be a unit vector of length 1.



This idea can be used to determine how a ray will reflect off a sphere and off a cylinder.

Example A sphere of radius 5, centred at $(0, 0, 0)$ is defined by $x^2 + y^2 + z^2 = 25$. The normal vector to the surface at the point $(5, 0, 0)$ is given as

$$\vec{n} = (5, 0, 0)$$

or $\vec{n} = (1, 0, 0)$ as a unit normal vector.

Example A sphere of radius 5, centred at $(0, 0, 0)$ is defined by $x^2 + y^2 + z^2 = 25$. The normal vector to the surface at the point $(0, 3, 4)$ is given as

$$\vec{n} = (0, 3, 4)$$

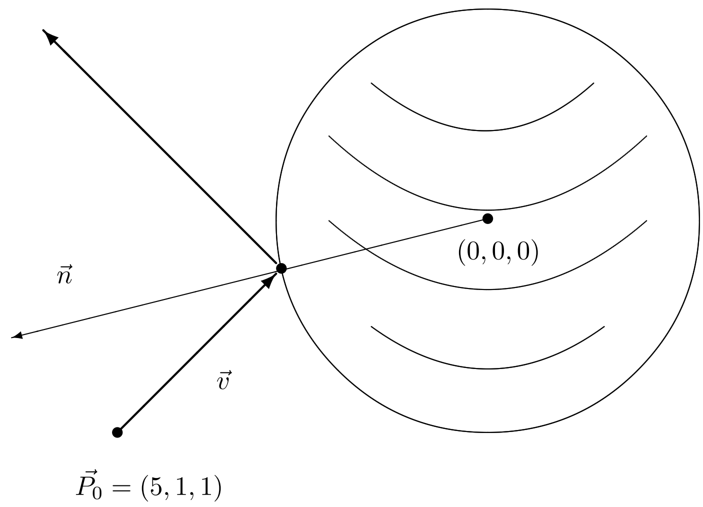
or $\vec{n} = (0, 0 \cdot 6, 0 \cdot 8)$ as a unit normal vector.

Once the normal vector is calculated in this way, it is now possible to calculate how a ray of light will be reflected off any circular surface.

Example A sphere with a reflective surface is defined by the equation

$$x^2 + y^2 + z^2 = 16$$

A light source at $(5, 1, 1)$ with a very narrow beam is turned on and points in the direction $\vec{v} = (-1, -1, -1)$. Find where the centre of the beam of light will hit the sphere and the direction of the reflected light.



The equation of the beam of light is

$$\vec{P} = \vec{P_0} + t.\vec{v}$$

i.e.,

$$\begin{aligned} \vec{P} &= (5, 1, 1) + t(-1, -1, -1) \\ &= (5 - t, 1 - t, 1 - t) \end{aligned}$$

This ray will intersect the sphere when

$$(5 - t)^2 + (1 - t)^2 + (1 - t)^2 = 16$$

i.e.,

$$3t^2 - 14t + 11 = 0$$

which has solution $t = 1$ and $t = 11/3$, using the quadratic formula. The parametric equations for the ray is $\vec{P} = (5, 1, 1) + t(-1, -1, -1)$, hence the two points of intersection are

$$\begin{aligned} \vec{P}(1) &= (5,1,1) + 1(-1,-1,-1) = (4,0,0) \\ \vec{P}(11/3) &= (5,1,1) + 11/3(-1,-1,-1) = (4/3, -8/3, -8/3) \end{aligned}$$

Since the ray starts at the point $(5, 1, 1)$ it will hit the sphere at the point $(4, 0, 0)$ first.

To find the reflected ray from the point $(4, 0, 0)$ we use the equation

$$\vec{R} = 2(\vec{n}.\vec{L})\vec{n} - \vec{L}$$

where the unit direction vector $-\vec{v}$, **towards the light** is

$$\vec{L} = \frac{(1,1,1)}{||\vec{v}||} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

and the unit normal vector is

$$\vec{n} = \frac{(4,0,0)}{4} = (1,0,0)$$

The reflected beam is in the direction

$$\vec{R} = 2(\vec{n}.\vec{L})\vec{n} - \vec{L}$$

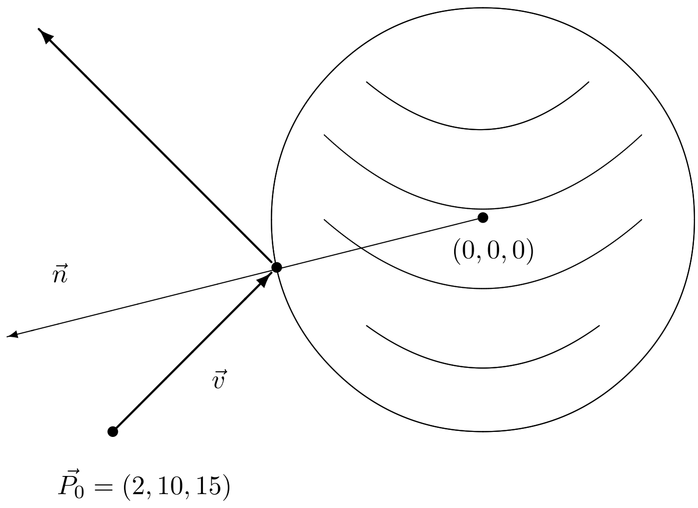
So

$$\begin{aligned} \vec{R} &= \frac{2}{\sqrt{3}}(1,0,0) - \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\ \vec{R} &= \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) \end{aligned}$$

Example A sphere with a reflective surface is defined by the equation

$$x^2 + y^2 + z^2 = 57$$

A light source at $(2, 10, 15)$ with a very narrow beam is turned on and points in the direction $\vec{v} = (-1, -3, -2)$. Find where the centre of the beam of light will hit the sphere and the direction of the reflected light.



The equation of the beam of light is

$$\vec{P} = \vec{P_0} + t.\vec{v}$$

i.e.,

$$\begin{aligned} \vec{P} &= (2, 10, 15) + t(-1, -3, -2) \\ &= (2 - t, 10 - 3t, 15 - 2t) \end{aligned}$$

This ray will intersect the sphere when

$$(2 - t)^2 + (10 - 3t)^2 + (15 - 2t)^2 = 57$$

i.e.,

$$7t^2 - 62t + 136 = 0$$

which has solution $t = 4 \cdot 85$ and $t = 4$, using the quadratic formula. The parametric equations for the ray is $\vec{P} = (2, 10, 15) + t(-1, -3, -2)$, hence the two points of intersection are

$$\begin{aligned} \vec{P}(4 \cdot 85) &= (2, 10, 15) + 4 \cdot 85(-1, -3, -2) \;=\; (-2 \cdot 85, -4.55, 5.3) \\ \vec{P}(4) &= (2, 10, 15) + 4(-1, -3, -2) \;=\; (-2, -2, 7) \end{aligned}$$

Since the ray starts at the point $(-2, -2, 7)$ it will hit the sphere at the point $(2, 10, 15)$ first.

To find the reflected ray from the point $(-2, -2, 7)$ we use the equation

$$\vec{R} \;=\; 2(\vec{n}.\vec{L})\vec{n} - \vec{L}$$

where the unit direction vector $-\vec{v}$, **towards the light** is

$$\vec{L} = \frac{(1,3,2)}{||\vec{v}||} = \left(\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right)$$

and the unit normal vector is

$$\vec{n} = \frac{(-2,-2,7)}{||\vec{v}||} = \left(\frac{-2}{\sqrt{57}}, \frac{-2}{\sqrt{57}}, \frac{7}{\sqrt{57}}\right)$$

The reflected beam is in the direction

$$\vec{R} \;=\; 2(\vec{n}.\vec{L})\vec{n} - \vec{L}$$

So

$$\vec{R} \;=\; \frac{12}{\sqrt{14}.\sqrt{57}}\left(\frac{-2}{\sqrt{57}}, \frac{-2}{\sqrt{57}}, \frac{7}{\sqrt{57}}\right) - \left(\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right)$$

$$\vec{R} \;=\; (-0 \cdot 379, -0 \cdot 379, -0 \cdot 141)$$