INSTRUCTIONS

Full marks will be awarded for the correct solutions to **ANY FIVE QUESTIONS**. This paper will be marked out of a TOTAL MAXIMUM MARK OF 100. Credit will be given for clearly presented solutions. **MATHEMATICAL TABLES**, if required, are available from the invigilator. Take note of the **USEFUL INFORMATION** presented with this examination paper.

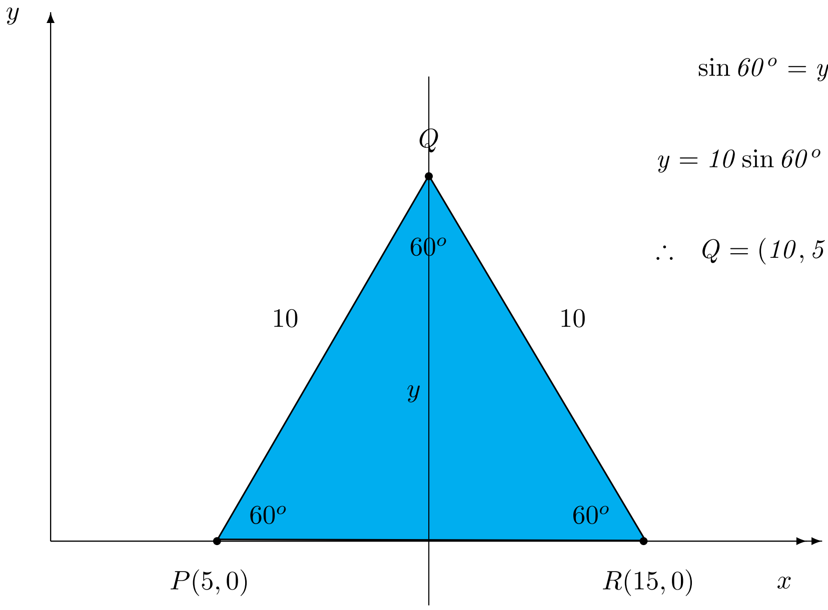
CW_ KCCGD_ B

BSc (Hons) in Computer Games Development

YEAR 1

AUTUMN, 2019

1. (a)



$$\sin 60^o = y/10$$

$$y = 10 \sin 60^o = 5\sqrt{3}$$

$$\therefore \quad Q = (10, 5\sqrt{3})$$

(b) Now

$$P(3,2,-1) \quad , \quad Q(6,6,0) \quad , \quad R(4,0,5)$$

i

$$P - 2Q + 3R = (3,2,-1) - 2(6,6,0) + 3(4,0,5) = (3,-10,14)$$

ii

$$\begin{aligned} 2P - 4Q &= 2(3,2,1) - 4(6,6,0) = (-18,-20,2) \\ ||2P - 4Q|| &= \sqrt{324 + 400 + 4} = \sqrt{728} \end{aligned}$$

iii

$$P.Q = 3.6 + 2.6 + (-1).0 = 30$$

iv

$$P \times Q = det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -1 \\ 6 & 6 & 0 \end{vmatrix} = (6,-6,6)$$

v

$$\begin{aligned} \vec{PQ} &= \vec{Q} - \vec{P} = (6,6,0) - (3,2,-1) = (3,4,1) \\ \vec{PR} &= \vec{R} - \vec{P} = (4,0,5) - (3,2,-1) = (1,-2,6) \\ \vec{RQ} &= \vec{Q} - \vec{R} = (6,6,0) - (4,0,5) = (2,6,-5) \end{aligned}$$

$$\vec{PQ} \times \vec{PR} = det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 1 \\ 1 & -2 & 6 \end{vmatrix} = (26,-17,-10)$$

$$\vec{RQ}.(\vec{PQ} \times \vec{PR}) = 2(26) + 6(-17) - 5(-10) = 52 - 102 + 50 = 0$$

The vectors defining this triangle $\triangle PQR$ are coplanar.

vi To determine the angle α

$$\cos\alpha = \frac{\vec{PQ}.\vec{PR}}{\|\vec{PQ}\|\|\vec{PR}\|} = \frac{1}{\sqrt{26}.\sqrt{41}} = 0 \cdot 0306 \quad , \quad \alpha = 88 \cdot 2^o$$

To determine the angle β

$$\cos\beta = \frac{\vec{RP}.\vec{RQ}}{\|\vec{RP}\|\|\vec{RQ}\|} = \frac{40}{\sqrt{41}.\sqrt{65}} = 0 \cdot 7748 \quad , \quad \beta = 39 \cdot 2^o$$

Finally $\alpha + \beta + \gamma = 180^o$, hence $\gamma = 52 \cdot 6^o$.

vii Hence the area of the triangle $\triangle PQR$ is

$$\frac{1}{2}\|\vec{PQ} \times \vec{PR}\|$$

where $\vec{PQ} \times \vec{PR} = (26, -17, -10)$. Now

$$\frac{1}{2}\|\vec{PQ} \times \vec{PR}\| = \frac{1}{2}\sqrt{676 + 289 + 100} = \frac{1}{2}\sqrt{1354}$$

The area of the triangle $\triangle PQR$ is $18 \cdot 4$ *sq. units*.

2. Let $P_0 = (-1, -1, -1)$. The line Ω goes through the points $A(1, 2, \frac{1}{2})$ and $B(-1, -1, -1)$, hence

$$\begin{aligned} \vec{v} &= \overrightarrow{AB} \\ &= \vec{B} - \vec{A} \\ &= (-1, -1, -1) - (1, 2, \frac{1}{2}) \\ &= (-2, -3, -\frac{3}{2}) \end{aligned}$$

Now

$$\begin{aligned} \overrightarrow{P_0P} &= \vec{P} - \vec{P_0} \\ &= (x, y, z) - (-1, -1, -1) \\ &= (x + 1, y + 1, z + 1) \end{aligned}$$

Now $\overrightarrow{P_0P} = t\vec{v}$, hence

$$\begin{aligned} (x + 1, y + 1, z + 1) &= t(-2, -3, -\frac{3}{2}) \\ &= (-2t, -3t, -\frac{3}{2}t) \end{aligned}$$

for some $t \in \mathbb{R}$. Expanding gives the parametric equations of the line Ω.

$$\begin{aligned} x &= -1 - 2t \\ y &= -1 - 3t \\ z &= -1 - \frac{3}{2}t \end{aligned}$$

Let $P_0 = (-2, 1, 2)$ and $\vec{n} = (-\frac{1}{4}, 1, 2)$.

(Shorthand Method) The equation of the plane Π is

$$-\frac{1}{4}x + y + 2z + d = 0$$

But $P_0 \in \Pi$, hence

$$-\frac{1}{4}(-2) + 1 + 2(2) + d = 0$$

Therefore $d = -\frac{11}{2}$. Finally the equation of the plane Π is

$$-\frac{1}{4}x + y + 2z - \frac{11}{2} = 0$$

i.e. $x - 4y - 8z + 22 = 0$. Now to calculate the point of intersection of the line Ω and the plane Π.

$$\begin{aligned} (-1-2t) - 4(-1-3t) - 8(-1-\frac{3}{2}t) + 22 &= 0 \\ -1 - 2t + 4 + 12t + 8 + 12t + 22 &= 0 \\ 22t &= -33 \\ t &= -\frac{3}{2} \end{aligned}$$

Hence, the point of intersection is

$$\begin{aligned} x &= -1 - 2(-\frac{3}{2}) = 2 \\ y &= -1 - 3(-\frac{3}{2}) = \frac{7}{2} \\ z &= -1 - \frac{3}{2}(-\frac{3}{2}) = \frac{5}{4} \end{aligned}$$

Therefore, $P = (2, \frac{7}{2}, \frac{5}{4})$.

3. (a) (Shorthand Method) The equation of the plane Π is

$$x + 9y + 8z + d = 0$$

But $P_0 = (1,1,4) \in \Pi$, hence $1 + 9(1) + 8(4) + d = 0$ Therefore $d = -42$. Finally the equation of the plane Π is

$$x + 9y + 8z - 42 = 0$$

(b) To find the parametric equations of the line of intersection Ω of the planes

$$\begin{aligned} \Gamma &: 2x - 2y + 4z - 6 = 0 \\ \Psi &: -4x - 2y - z - 2 = 0 \end{aligned}$$

Let $\vec{n_\Gamma} = (2,-2,4)$, $\vec{n_\Psi} = (-4,-2,-1)$.

$$\vec{n_\Gamma} \times \vec{n_\Psi} = det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 4 \\ -4 & -2 & -1 \end{vmatrix} = (10,-14,-12)$$

The line Ω is parallel to $\vec{v} = (10,-14,-12)$.
We require a common point P_0. Now $P_0 = (4,-7,-4)$.

$$\begin{aligned} \overrightarrow{P_0P} &= \vec{P} - \vec{P_0} \\ &= (x,y,z) - (4,-7,-4) \\ &= (x-4, y+7, z+4) \end{aligned}$$

Now $\overrightarrow{P_0P} = t\vec{v}$, hence

$$\begin{aligned}(x-4, y+7, z+4) &= t(10, -14, -12) \\ &= (10t, -14t, -12t)\end{aligned}$$

for some $t \in \mathbb{R}$. Expanding gives the parametric equations of the line Ω.

$$\begin{aligned}x &= 4 + 10t \\ y &= -7 - 14t \\ z &= -4 - 12t\end{aligned}$$

4. (a) Let

$$A.B = \frac{1}{2}\begin{pmatrix} -4 & 8 & 2 \\ -3 & 5 & 3 \\ -2 & 4 & 0 \end{pmatrix}\begin{pmatrix} 6 & -4 & -7 \\ 3 & -2 & -3 \\ 1 & 0 & -2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A is *invertible.*

It has an *inverse* given as B.

(b) i. We form the following matrix

$$\left(\begin{array}{ccc|ccc} 6 & -4 & -7 & 1 & 0 & 0 \\ 3 & -2 & -3 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 \end{array}\right)$$

Now

$$\underline{R_1 \leftrightarrow R_3}\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & 1 \\ 3 & -2 & -3 & 0 & 1 & 0 \\ 6 & -4 & -7 & 1 & 0 & 0 \end{array}\right)$$

$$\frac{R_2 - 3R_1}{R_3 - 6R_1}\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & 1 \\ 0 & -2 & 3 & 0 & 1 & -3 \\ 0 & -4 & 5 & 0 & 0 & -6 \end{array}\right)$$

$$\underline{R_3 - 2R_2}\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & 1 \\ 0 & -2 & 3 & 0 & 1 & -3 \\ 0 & 0 & -1 & 1 & -2 & 0 \end{array}\right)$$

$$\frac{R_2 + 3R_3}{R_1 - 2R_3}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 4 & 1 \\ 0 & -2 & 0 & 3 & -5 & -3 \\ 0 & 0 & -1 & 1 & -2 & 0 \end{array}\right)$$

$$\begin{array}{c}(-\frac{1}{2})R_2 \\ \hline (-1)R_3\end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 4 & 1 \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & 1 & -1 & 2 & 0 \end{array}\right)$$

Therefore, we have

$$A^{-1} = \frac{1}{2}\begin{pmatrix} -4 & 8 & 2 \\ -3 & 5 & 3 \\ -2 & 4 & 0 \end{pmatrix}$$

ii. To find a matrix B such that $BA = C$ where

$$C = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix}$$

note that we can use the inverse A^{-1} as follows

$$\begin{array}{rcl} BAA^{-1} & = & CA^{-1} \\ \Rightarrow \quad BI & = & CA^{-1} \\ \therefore \quad B & = & CA^{-1} \end{array}$$

hence

$$B = CA^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} -4 & 8 & 2 \\ -3 & 5 & 3 \\ -2 & 4 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -8 & 16 & 2 \\ -3 & 7 & -3 \end{pmatrix}$$

5. (a) Firstly

$$\begin{array}{rcccccc} \vec{PQ} & = & \vec{Q} - \vec{P} & = & (3,4,1) - (1,2,5) & = & (2,2,-4) \\ \vec{PR} & = & \vec{R} - \vec{P} & = & (4,0,6) - (1,2,5) & = & (3,-2,1) \\ \vec{RQ} & = & \vec{Q} - \vec{R} & = & (3,4,1) - (4,0,6) & = & (-1,4,-5) \end{array}$$

$$\vec{PQ} \times \vec{PR} = det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & -4 \\ 3 & -2 & 1 \end{vmatrix} = (-6, -14, -10)$$

The area of the triangle $\triangle PQR$ is

$$\frac{1}{2}\|\vec{PQ} \times \vec{PR}\|$$

where $\vec{PQ} \times \vec{PR} = (-6, -14, -10)$. Now

$$\frac{1}{2}\|\vec{PQ} \times \vec{PR}\| = \frac{1}{2}\sqrt{36 + 196 + 100} = \frac{1}{2}\sqrt{332}$$

The area of the triangle $\triangle PQR$ is $9 \cdot 11$ *sq. units*

(b)

$$T[P] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}$$

$$T[Q] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

$$T[R] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 0 \end{pmatrix}$$

(c) To show that A is orthogonal, note that a square matrix A is an *orthogonal* matrix if $A^{-1} = A^t$. Pre-multiplying both sides by A yield

$$A.A^{-1} = I = A.A^t$$

Now

$$A.A^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

When an orthogonal matrix is used to rotate vectors, it will keep the lengths of the vectors preserved as will the angle between the vectors. So any orthogonal matrix may represent a rotation.

6. (a) The equation of the beam of light is

$$\vec{P} = \vec{P_0} + t.\vec{v}$$

$$\begin{aligned} \vec{P} &= (5,1,1) + t(-1,-1,-1) \\ &= (5-t, 1-t, 1-t) \end{aligned}$$

This ray will intersect the sphere when

$$(5-t)^2 + (1-t)^2 + (1-t)^2 = 16$$

$$3t^2 - 14t + 11 = 0$$

which has solution $t = 1$ and $t = 11/3$, using the quadratic formula. The parametric equations for the ray is $\vec{P} = (5,1,1) + t(-1,-1,-1)$, hence the two points of intersection are

$$\begin{aligned} \vec{P}(1) &= (5,1,1) + 1(-1,-1,-1) = (4,0,0) \\ \vec{P}(11/3) &= (5,1,1) + 11/3(-1,-1,-1) = (4/3, -8/3, -8/3) \end{aligned}$$

Since the ray starts at the point $(5,1,1)$ it will hit the sphere at the point $(4,0,0)$ first.

(b) To find the reflected ray from the point $(4,0,0)$ we use the equation

$$\vec{R} \;=\; 2(\vec{n}.\vec{L})\vec{n} - \vec{L}$$

where the unit direction vector $-\vec{v}$, **towards the light** is

$$\vec{L} = \frac{(1,1,1)}{||\vec{v}||} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

and the unit normal vector is

$$\vec{n} = \frac{(4,0,0)}{4} = (1,0,0)$$

The reflected beam is in the direction

$$\vec{R} \;=\; 2(\vec{n}.\vec{L})\vec{n} - \vec{L}$$

So

$$\vec{R} \;=\; \frac{2}{\sqrt{3}}(1,0,0) - \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\vec{R} \;=\; \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

7. (a) Given the *quaternions*

$$\begin{aligned} q_1 &= 1 + 4i + 2j - k \\ q_2 &= 2 - 4i + 5j + 2k \end{aligned}$$

i

$$\begin{aligned} 3q_1 + q_2 &= 3(1 + 4i + 2j - k) + (2 - 4i + 5j + 2k) \\ &= (3+2) + (12-4)i + (6+5)j + (-3+2)k \\ &= 5 + 8i + 11j - k \end{aligned}$$

ii

$$\begin{aligned} q_1 \times q_2 &= (1 + 4i + 2j - k) \times (2 - 4i + 5j + 2k) \\ &= (2 - 4i + 5j + 2k) + (8i - 16i^2 + 20ij + 8ik) \\ &\qquad + (4j - 8ji + 10j^2 + 4jk) + (-2k + 4ki - 5kj - 2k^2) \\ &= (2 + 16 - 10 + 2) - 4i + 5j + 2k + 8i + 20k - 8j + 4j + 8k + 4i - 2k - 4j + 5i \\ &= 10 + 13i - 3j + 28k \end{aligned}$$

iii Now $\bar{q_1} = 1 - 4i - 2j + k$ and $|q_1|^2 = 1^2 + 4^2 + (-2)^2 + (-1)^2 = 22$. Hence

$$q_1^{-1} = \frac{1}{22}(1 - 4i - 2j + k)$$

(b) Rotate the point $r(2, 2, 2)$ 180^o *anti-clockwise* about the y-axis, (i.e., the vector $\vec{v} = (0, 1, 0)$) using a quaternion.

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(li + mj + nk) = \cos\frac{180}{2} + \sin\frac{180}{2}(0i + j + 0k)$$

Hence $q = j$. Also $q^{-1} = -j$ and $r = 2i + 2j + 2k$. Now

$$\begin{aligned} Rotation &= q.r.q^{-1} \\ &= j.(2i + 2j + 2k). - j \\ &= (2ji + 2j^2 + 2jk). - j \\ &= (-2 + 2i - 2k). - j \\ &= 2j - 2ij + 2kj \\ Rotation &= -2i + 2j - 2k \end{aligned}$$

Hence

$$r(2, 2, 2) \longrightarrow r'(-2, 2, -2)$$