INSTRUCTIONS

Full marks will be awarded for the correct solutions to **ANY FIVE QUESTIONS**. This paper will be marked out of a TOTAL MAXIMUM MARK OF 100. Credit will be given for clearly presented solutions. **MATHEMATICAL TABLES**, if required, are available from the invigilator. Take note of the **USEFUL INFORMATION** presented with this examination paper.

CW_ KCCGD_ B

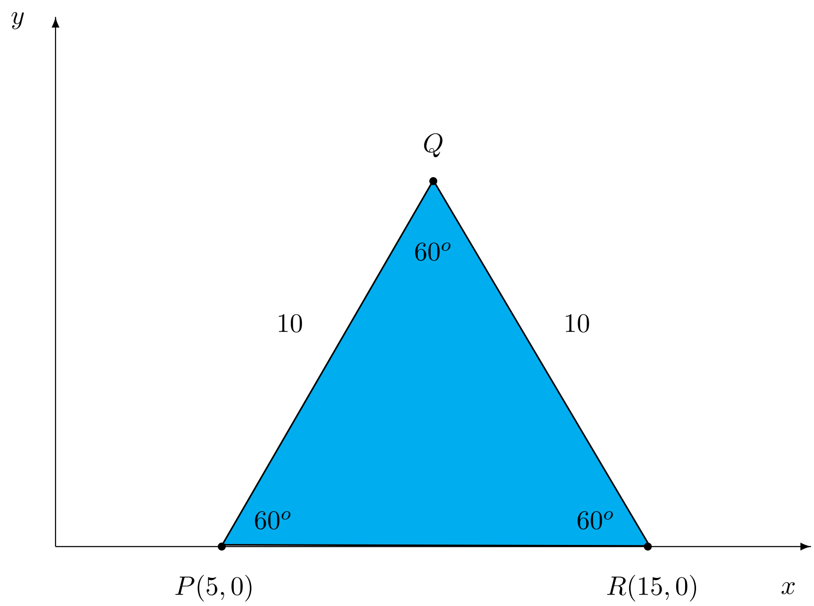
BSc (Hons) in Computer Games Development

YEAR 1

AUTUMN, 2019

1. (a) This coordinate diagrams in $\mathbb{R}^2$ shows an equilateral triangle PQR.

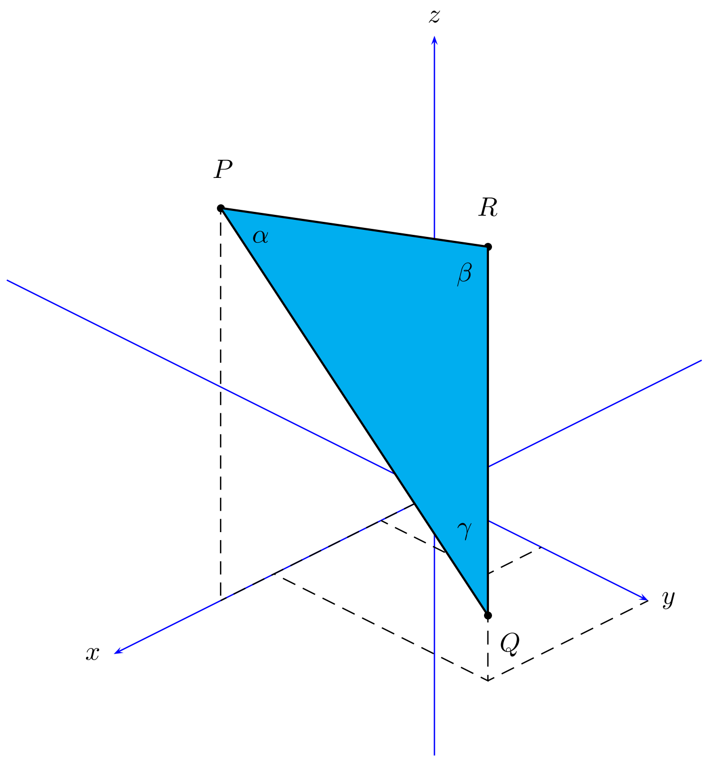
Using *basic trigonometry*, or otherwise, determine the coordinate Q.



4 marks

(b) Consider the following points in $\mathbb{R}^3$.

$$P(3,2,-1) \quad , \quad Q(6,6,0) \quad , \quad R(4,0,5)$$



i Evaluate $P - 2Q + 3R$.

ii Determine $||2P - 4Q||$.

iii Evaluate $P.Q$, i.e., the *scalar product* of P with Q.

iv Evaluate $P \times Q$.

v Show that the vectors defining this triangle $\triangle PQR$ are *coplanar*.

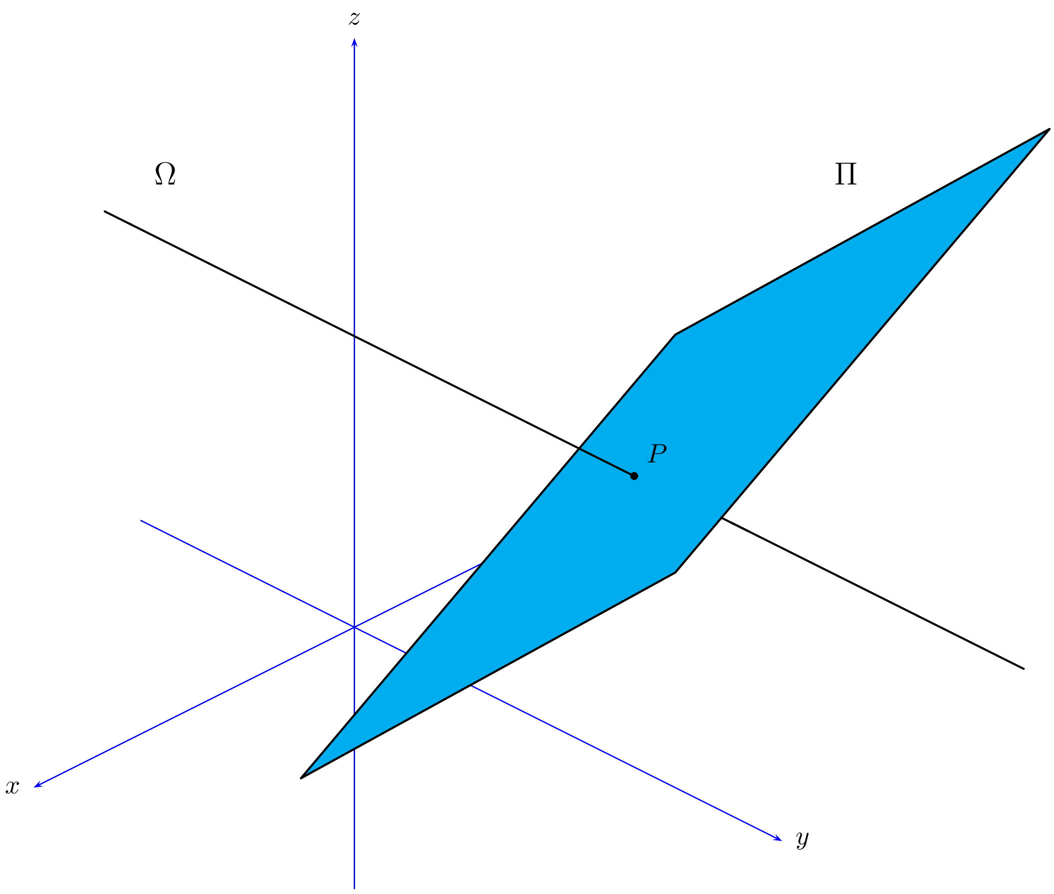
vi Determine all internal angles α, β, γ of the triangle $\triangle PQR$.

vii Determine the area of the triangle $\triangle PQR$.

16 marks

2. The straight line Ω goes through the point $(1, 2, 1/2)$ and $(-1, -1, -1)$.
The vector $(-1/4, 1, 2)$ is perpendicular to the plane Π, and Π passes through $(-2, 1, 2)$.
Find the point P at which Ω meets Π.

20 marks



3. (a) Find a *point-normal form* of the equation of the plane containing the point $P_0 = (1, 1, 4)$ with normal vector $\vec{n} = (1, 9, 8)$.

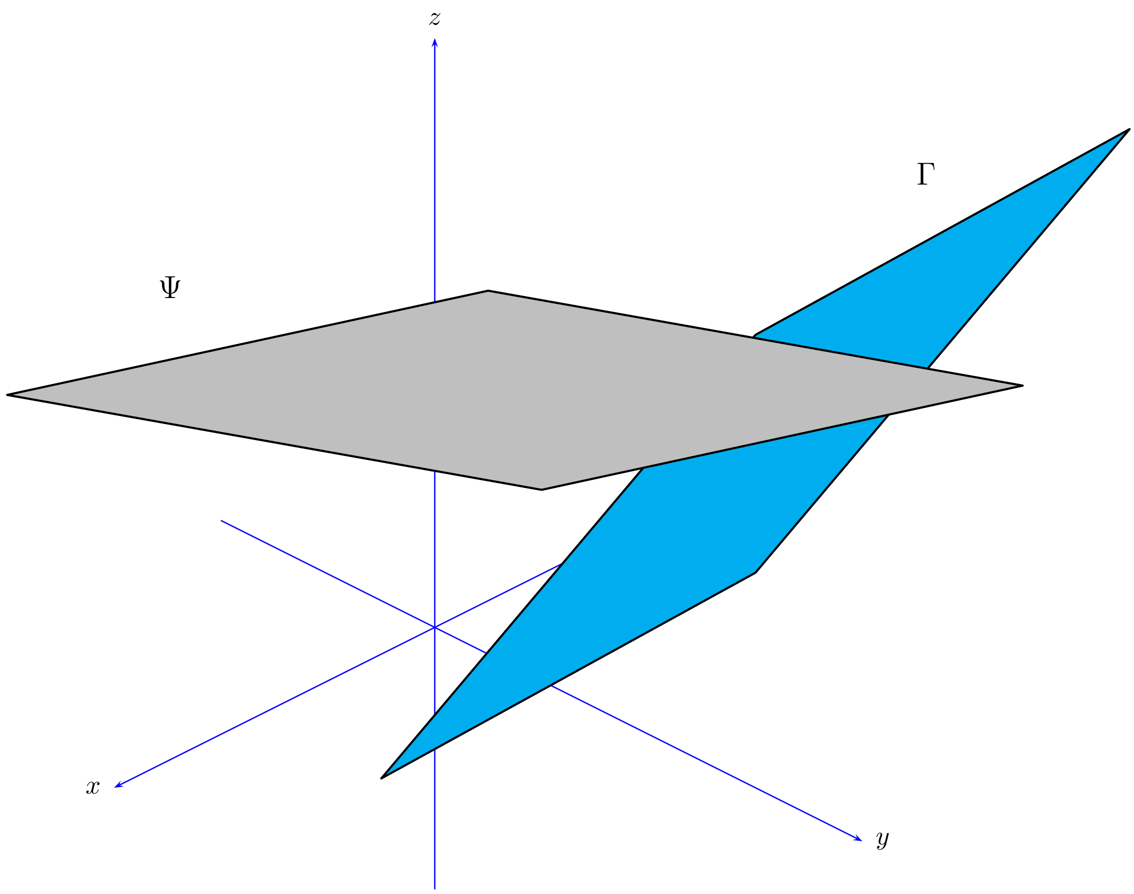
6 marks

(b) Find the parametric equations of the line of intersection Ω of the planes

$$\begin{aligned} \Gamma &: \quad 2x - 2y + 4z - 6 = 0 \\ \Psi &: \quad -4x - 2y - z - 2 = 0 \end{aligned}$$

Note that $(4, -7, -4) \in \Omega$.

14 marks



4. (a) Let

$$A = \frac{1}{2}\begin{pmatrix} -4 & 8 & 2 \\ -3 & 5 & 3 \\ -2 & 4 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 6 & -4 & -7 \\ 3 & -2 & -3 \\ 1 & 0 & -2 \end{pmatrix}$$

Determine $A.B$. What comment can you make about the matrix A?

6 marks

(b) Let

$$A = \begin{pmatrix} 6 & -4 & -7 \\ 3 & -2 & -3 \\ 1 & 0 & -2 \end{pmatrix}$$

i By a sequence of *row operations* find A^{-1}, the *inverse* of A.

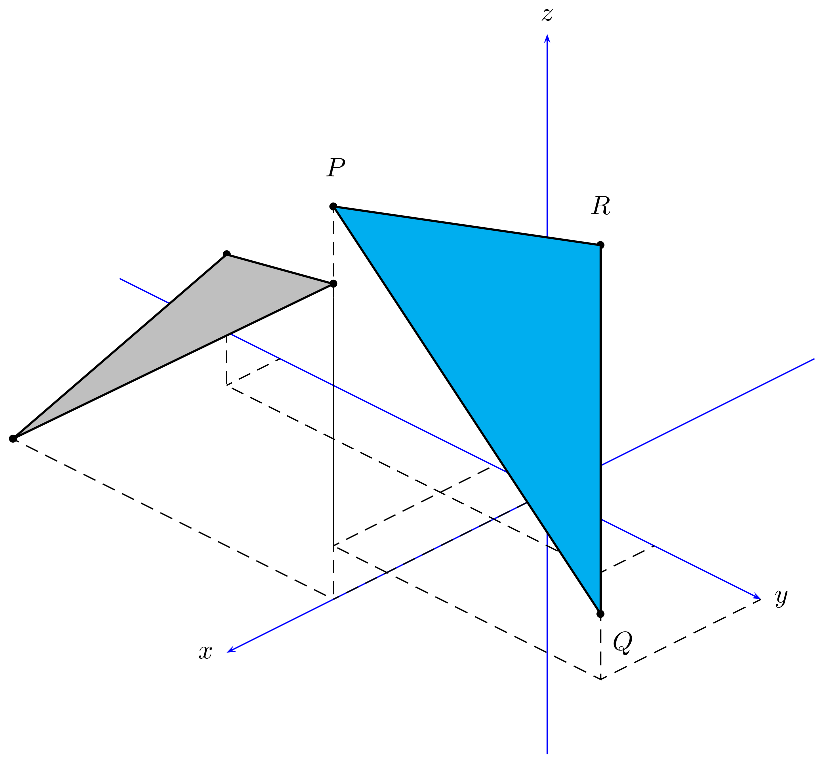
ii Find a matrix B such that

$$BA = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix}$$

14 marks

5. Consider the following points in $\mathbb{R}^3$.

$$P(1,2,5) \quad , \quad Q(3,4,1) \quad , \quad R(4,0,6)$$



(a) Find the area of a triangle PQR.

10 marks

(b) Let $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that

$$T(x) = [A]x$$

denote a *matrix transformation* with *standard matrix* A. Rotate the triangle PQR *anti-clockwise* about the positive x-axis through an angle of 90^o where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

6 marks

(c) Show that the *standard matrix* from part (b) is *orthogonal*, i.e., $A^{-1} = A^t$.

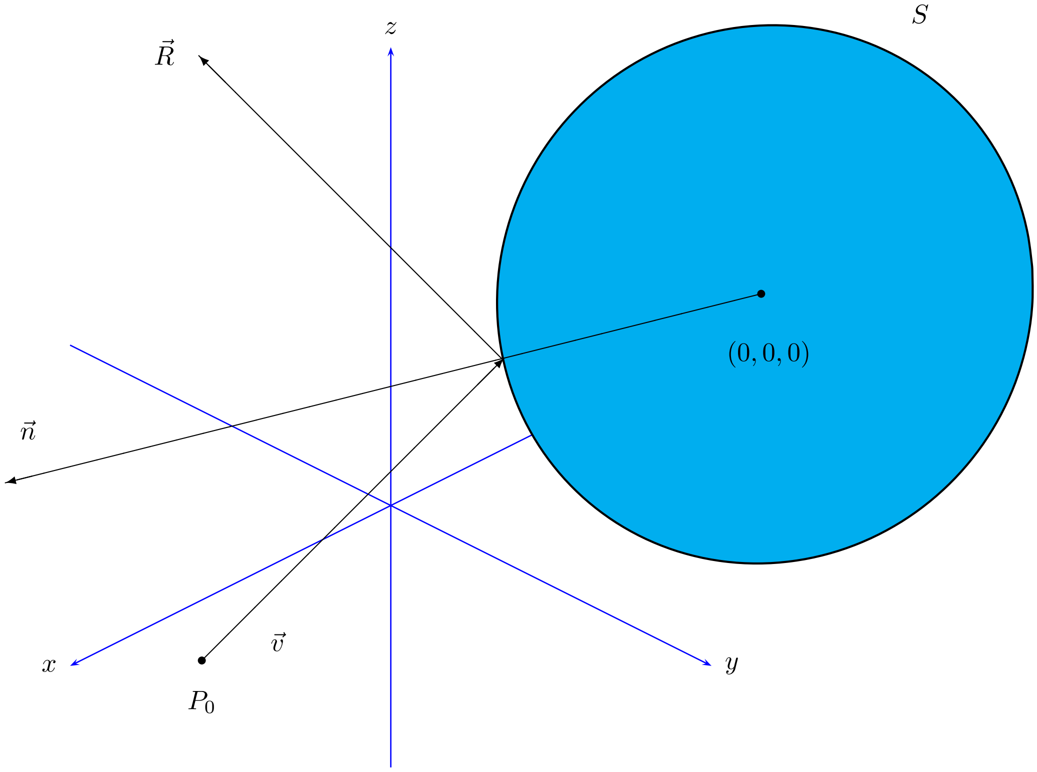
Comment on the important property of orthogonal matrices when applied to the rotation of the triangle PQR.

4 marks

6. A sphere S with a reflective surface is defined by the equation

$$x^2 + y^2 + z^2 = 16$$

A light source at $P_0 = (5, 1, 1)$ with a very narrow beam is turned on and points in the direction $\vec{v} = (-1, -1, -1)$.



(a) If the equation of the beam of light is

$$\vec{P} = \vec{P}_0 + t.\vec{v}$$

find where the centre of this beam will hit the sphere S.

10 marks

(b) Find the direction of the reflected beam $\vec{R}$.

10 marks

7. (a) Given the *quaternions*

$$\begin{aligned} q_1 &= 1 + 4i + 2j - k \\ q_2 &= 2 - 4i + 5j + 2k \end{aligned}$$

Evaluate each of the following

i $3q_1 + q_2$,

ii $q_1 \times q_2$,

iii q_1^{-1}, the *inverse* of q_1.

12 marks

(b) Consider the point $r(2, 2, 2)$ in $\mathbb{R}^3$.

Rotate this point 180^o *anti-clockwise* about the y-axis, (i.e., the vector $\vec{v} = (0, 1, 0)$) using a quaternion.

Note: Full workings must be shown for this question.

8 marks

USEFUL INFORMATION

VECTORS IN $\mathbb{R}^3$:

For any vector $\vec{u} = (u_1, u_2, u_3)$ in $\mathbb{R}^3$, we define the magnitude of $\vec{u}$ to be the non-negative real number

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Suppose that $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$ and that $\theta \in [0, \pi]$ represents the angle between them. We define the scalar product $\vec{u}.\vec{v}$ of $\vec{u}$ and $\vec{v}$ by $\vec{u}.\vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$. Hence

$$\cos\theta = \frac{\vec{u}.\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

Alternatively, we can write $\vec{u}.\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$.

Theorem 1 *Suppose that $\vec{u}, \vec{v} \in \mathbb{R}^3$. Then the parallelogram with $\vec{u}$ and $\vec{v}$ as two of its sides has area $\|\vec{u} \times \vec{v}\|$.*

Theorem 2 *Suppose that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$. The vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ are termed coplanar if*

$$\vec{u}.(\vec{v} \times \vec{w}) = 0$$

Theorem 3 *The perpendicular distance D of a point $P = (x_0, y_0, z_0)$ from the plane $ax + by + cz + d = 0$ is given by*

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

LINES AND PLANES IN $\mathbb{R}^3$:

Let Ω be a **line** in $\mathbb{R}^3$.

i Find a point $P_0 = (x_0, y_0, z_0)$ which is on Ω.

ii Find a vector $\vec{v} = (v_1, v_2, v_3)$ which is parallel to Ω.

Then $t\vec{v}$ is also parallel to Ω and the line is the set of all points $P = (x, y, z)$ for which the vector

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

is equal to $t\vec{v}$, or

$$\overrightarrow{P_0P} = t\vec{v}$$

for some $t \in \mathbb{R}$.

Let Π be a **plane** in $\mathbb{R}^3$.

i Find the co-ordinates of a point $P_0 = (x_0, y_0, z_0)$ which is in the plane.

ii Find a vector $\vec{n} = (a, b, c)$ perpendicular to the plane.

Then the plane Π consists of those points $P = (x, y, z)$ for which the vector

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

is orthogonal to $\vec{n}$, or

$$\overrightarrow{P_0P}.\vec{n} = 0$$

QUATERNIONS:

From the formula defining multiplication of quaternions we have that

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k \quad , \quad jk = -kj = i \quad , \quad ki = -ik = j$$

If follows directly from these identities that $ijk = -1$. The operation of multiplication on the set $\mathbb{H}$ of quaternions is **not** commutative.

QUATERNIONS AND ROTATIONS:

The following quaternion

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(li + mj + nk)$$

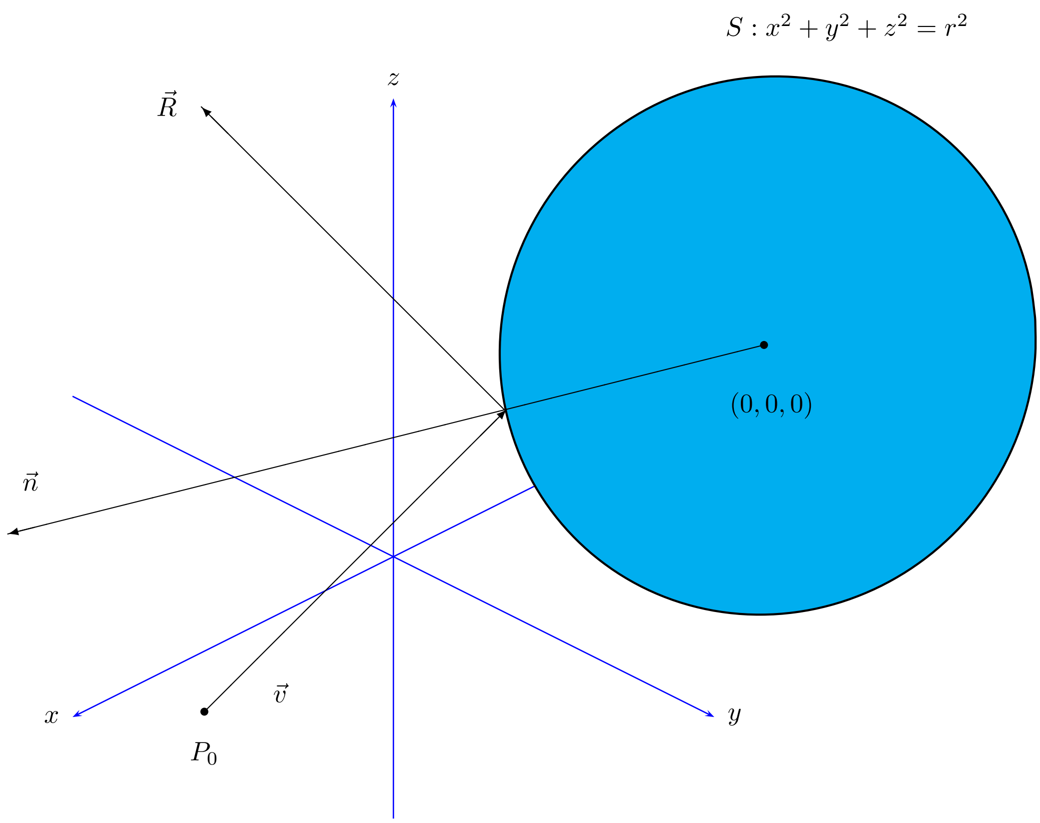
will rotate a point $r(x, y, z)$ through θ *anti-clockwise* about the unit vector $\vec{n} = (l, m, n)$. The rotation will be achieved by $\textbf{Rotation} = \mathbf{q}.\mathbf{r}.\mathbf{q^{-1}}$ where $r(x, y, z) = xi + yj + zk$.

Note also that for a quaternion $q = w + xi + yj + zk$

$$q^{-1} = \frac{1}{|q|^2}\bar{q}$$

the modulus is given as $|q|^2 = w^2 + x^2 + y^2 + z^2$ and the conjugate is given as $\bar{q} = w - xi - yj - zk$

RAY TRACING :



The equation of the beam of light (starting at P_0 and in the direction $\vec{v}$) is given as

$$\vec{P} = \vec{P_0} + t.\vec{v}$$

The vector $\vec{L}$ is a vector **pointing towards** the light source, i.e., $\vec{L} = -\vec{v}$ and the vector $\vec{R}$ represents the direction of the reflected light. If the normal to the surface is $\vec{n}$ and ensuring the vectors $\vec{L}$ and $\vec{n}$ are **unit vectors**, then the reflected beam is represented as

$$\vec{R} \;=\; 2(\vec{n}.\vec{L})\vec{n} - \vec{L}$$