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At the Heart of South Leinster

INSTRUCTIONS

Full marks will be awarded for the correct solutions to **ANY FIVE QUESTIONS**. This paper will be marked out of a TOTAL MAXIMUM MARK OF 100. Credit will be given for clearly presented solutions. **MATHEMATICAL TABLES**, if required, are available from the invigilator. Take note of the **USEFUL INFORMATION** presented with this examination paper.

CW_ KCCGD_ B

BSc (Hons) in Computer Games Development

YEAR 1

Summer, 2016

1. Consider the following vectors $\vec{u} = (0, 8, -1)$, $\vec{v} = (5, 5, 0)$ and $\vec{w} = (8, 0, 1)$ in $\mathbb{R}^3$.

i Evaluate $\vec{u} - 2\vec{v} + 3\vec{w}$

ii Determine $||2\vec{u} - 4\vec{v}||$

iii Evaluate $\vec{u}.\vec{v}$

iv Determine the angle θ between $\vec{u}$ and $\vec{v}$.

v Determine the *vector projection* of $\vec{u}$ along $\vec{v}$, i.e., $proj_{\vec{v}}(\vec{u})$.

vi Calculate the components a, b and c of a non-zero vector that is orthogonal to $\vec{u}$ and $\vec{v}$.

vii Determine the area of the *parallelogram* that is defined by $\vec{u}$ and $\vec{v}$.

viii Determine $\vec{u}.(\vec{v} \times \vec{w})$. What comment can you make about the vectors $\vec{u}, \vec{v}$ and $\vec{w}$?

20 marks

2. (a) Consider the following points in $\mathbb{R}^3$.

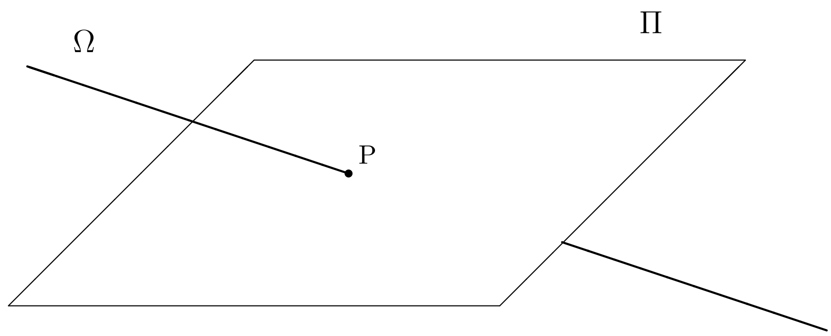
$$P(1,1,-1) \quad , \quad Q(1,-2,4) \quad , \quad R(2,2,0)$$

Find the equation of the plane Θ that contain these points.

8 marks

(b) The straight line Ω goes through the point $(1,2,1/2)$ and $(-1,-1,-1)$. The vector $(-1/4,1,2)$ is perpendicular to the plane Π, and Π passes through $(-2,1,2)$. Find the point P at which Ω meets Π.

12 marks



3. (a) Given the quaternions

$$\begin{aligned} q_1 &= 1+4i+6j+k \\ q_2 &= 2-4i+4j-4k \end{aligned}$$

Evaluate each of the following

i $q_1 \times q_2$

ii q_1^{-1}

8 marks

(b) We considered two methods to achieve a rotation in $\mathbb{R}^3$. Consider the point $r(-3,4,2)$. Rotate this point through 180^o *anti-clockwise* about the x-axis, (i.e., the vector $\vec{v}=(1,0,0)$) firstly using a quaternion. Repeat the exercise to achieve the same rotation using a matrix transformation.

Note: Full workings must be shown for this question.

12 marks

4. (a) Consider the following points in $\mathbb{R}^3$.

$$P(1,-2,4) \quad , \quad Q(2,0,-1) \quad , \quad R(-1,1,5)$$

Find the area of a triangle PQR.

8 marks

(b) Let $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that

$$T(x) = Ax$$

denote a *matrix transformation* with *standard matrix* A.

i Write the *standard matrix* A required to rotate a point *anti-clockwise* about the positive z-axis through an angle of 270^o.

ii Rotate the triangle PQR from part (a) using A.

iii Show that the *standard matrix* A is *orthogonal*, i.e., $A^{-1} = A^t$.

iv Comment on the important property of orthogonal matrices when applied to the rotation of the triangle PQR from part (a).

8 marks

(c) Let $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that

$$T(x) = Ax$$

denote a *matrix transformation* with *standard matrix* A. Find the *standard matrix* for the operator $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that first rotates a vector *anti-clockwise* about the y-axis through an angle 90^o, then rotates the resulting vector *anti-clockwise* about the x-axis through an angle of 180^o, and then dilates that vector by a factor of $k = 4$. **Note** that the standard matrix A for the operator is expressed by the composition

$$A = T_3 \circ T_2 \circ T_1 = [T_3].[T_2].[T_1]$$

4 marks

5. (a) Let

$$A = \begin{pmatrix} 1 & 2 \\ -5 & 2 \\ -5 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 5 \\ -3 & 2 & 1 \end{pmatrix}$$

Determine each of the following

i $3A + B^t$

ii $A.B$

5 marks

(b) Let

$$A = \begin{pmatrix} 3 & 5 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & 2 \end{pmatrix}$$

By a sequence of *row operations* find A^{-1}, the *inverse* of A. Use this inverse to find the solution of the following system of linear equations.

$$\begin{aligned} 3x + 5y - z &= 3 \\ x + z &= 5 \\ -x - y + 2z &= 4 \end{aligned}$$

15 marks

6. (a) Find the parametric equation for the line that starts at the point $\vec{P_0} = (1, 3, 2)$ and points in the direction $\vec{v} = (0, 0 \cdot 5, 0 \cdot 5)$.

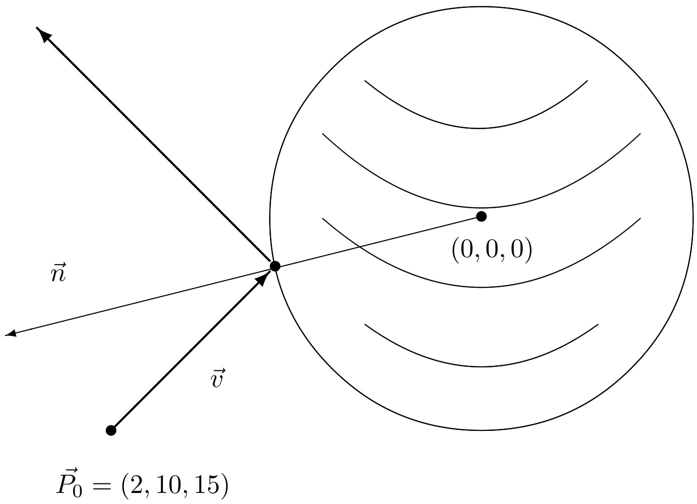
Calculate the position of three different points that lie on this line and draw a diagram that shows these three points on the line.

6 marks

(b) A sphere with a reflective surface is defined by the equation

$$x^2 + y^2 + z^2 = 57$$

A light source at $(2, 10, 15)$ with a very narrow beam is turned on and points in the direction $\vec{v} = (-1, -3, -2)$. Find where the centre of the beam of light will hit the sphere and the direction of the reflected light.



14 marks

USEFUL INFORMATION

VECTORS IN $\mathbb{R}^3$:

For any vector $\vec{u} = (u_1, u_2, u_3)$ in $\mathbb{R}^3$, we define the magnitude of $\vec{u}$ to be the non-negative real number

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Suppose that $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$ and that $\theta \in [0, \pi]$ represents the angle between them. We define the scalar product $\vec{u}.\vec{v}$ of $\vec{u}$ and $\vec{v}$ by $\vec{u}.\vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$. Hence

$$\cos\theta = \frac{\vec{u}.\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

Alternatively, we can write $\vec{u}.\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$.

The *vector projection* of $\vec{u}$ along $\vec{v}$ is defined as

$$proj_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u}.\vec{v}}{\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|}$$

Theorem 1

Suppose that $\vec{u}, \vec{v} \in \mathbb{R}^3$. Then the parallelogram with $\vec{u}$ and $\vec{v}$ as two of its sides has area $\|\vec{u} \times \vec{v}\|$.

Theorem 2

The perpendicular distance D *of a point* $P = (x_0, y_0, z_0)$ *from the plane* $ax+by+cz+d = 0$ *is given by*

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

LINES AND PLANES IN $\mathbb{R}^3$:

Let Ω be a **line** in $\mathbb{R}^3$.

i Find a point $P_0 = (x_0, y_0, z_0)$ which is on Ω.

ii Find a vector $\vec{v} = (v_1, v_2, v_3)$ which is parallel to Ω.

Then $t\vec{v}$ is also parallel to Ω and the line is the set of all points $P = (x, y, z)$ for which the vector

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

is equal to $t\vec{v}$, or

$$\overrightarrow{P_0P} = t\vec{v}$$

for some $t \in \mathbb{R}$.

Let Π be a **plane** in $\mathbb{R}^3$.

i Find the co-ordinates of a point $P_0 = (x_0, y_0, z_0)$ which is in the plane.

ii Find a vector $\vec{n} = (a, b, c)$ perpendicular to the plane.

Then the plane Π consists of those points $P = (x, y, z)$ for which the vector

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

is orthogonal to $\vec{n}$, or

$$\overrightarrow{P_0P}.\vec{n} = 0$$

QUATERNIONS:

From the formula defining multiplication of quaternions we have that

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k \quad , \quad jk = -kj = i \quad , \quad ki = -ik = j$$

If follows directly from these identities that $ijk = -1$. The operation of multiplication on the set $\mathbb{H}$ of quaternions is **not** commutative.

QUATERNIONS AND ROTATIONS:

The following quaternion

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(li + mj + nk)$$

will rotate a point $r(x, y, z)$ through θ *anti-clockwise* about the unit vector $\vec{n} = (l, m, n)$. The rotation will be achieved by **Rotation** $= \mathbf{q}.\mathbf{r}.\mathbf{q^{-1}}$ where $r(x, y, z) = xi + yj + zk$. Note also that for a quaternion $q = w + xi + yj + zk$

$$q^{-1} = \frac{1}{|q|^2}\bar{q}$$

the modulus is given as $|q|^2 = w^2 + x^2 + y^2 + z^2$ and the conjugate is given as $\bar{q} = w - xi - yj - zk$

MATRIX TRANSFORMATIONS IN $\mathbb{R}^3$:

The following are *standard matrices* in $\mathbb{R}^3$ required to

i *rotate a point anti-clockwise* through an angle of θ about the x-axis, y-axis and z-axis respectively

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$A = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ii *contract or dilate* with factor k

$$A = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

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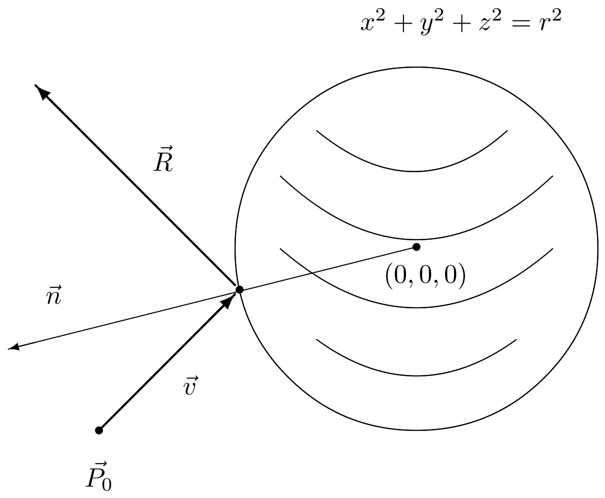
RAY TRACING :



The equation of the beam of light (starting at P_0 and in the direction $\vec{v}$) is given as

$$\vec{P} = \vec{P}_0 + t.\vec{v}$$

The vector $\vec{L}$ is a vector **pointing towards** the light source, i.e., $\vec{L} = -\vec{v}$ and the vector $\vec{R}$ represents the direction of the reflected light. If the normal to the surface is $\vec{n}$ and ensuring the vectors $\vec{L}$ and $\vec{n}$ are **unit vectors**, then the reflected beam is represented as

$$\vec{R} \; = \; 2(\vec{n}.\vec{L})\vec{n} - \vec{L}$$