INSTRUCTIONS

Full marks will be awarded for the correct solutions to **ANY FIVE QUESTIONS**. This paper will be marked out of a TOTAL MAXIMUM MARK OF 100. Credit will be given for clearly presented solutions. **MATHEMATICAL TABLES**, if required, are available from the invigilator. Take note of the **USEFUL INFORMATION** presented with this examination paper.

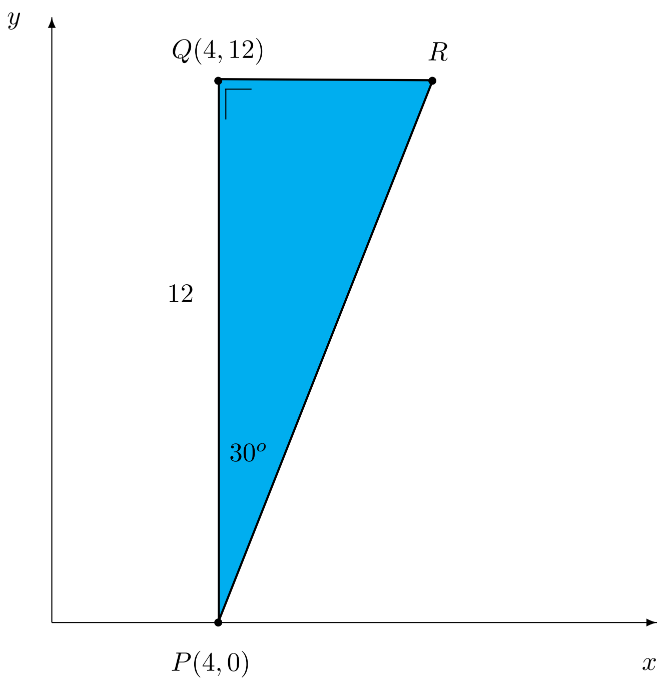
CW_ KCCGD_ B

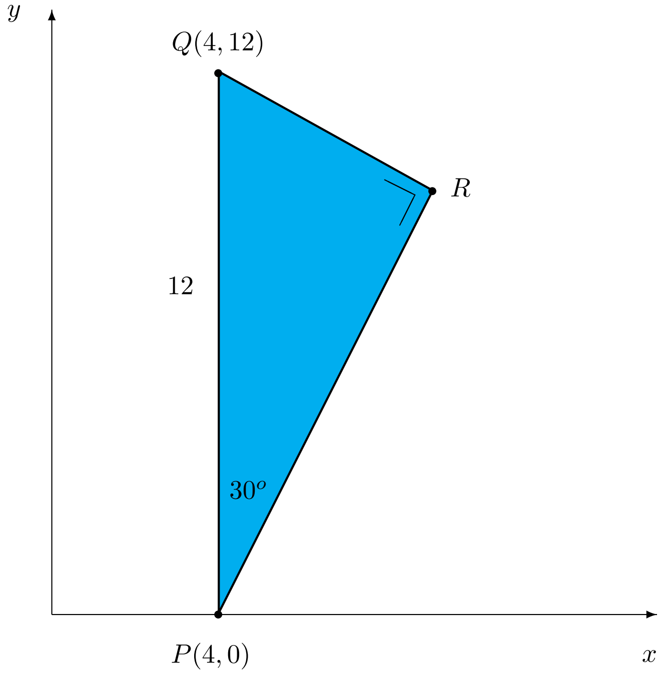
BSc (Hons) in Computer Games Development

YEAR 1

SUMMER, 2017

1. (a) In each of the following coordinate diagrams in $\mathbb{R}^2$ the triangle PQR is a right-angled triangle. In each case, using *basic trigonometry*, determine the coordinates of the point R.





6 marks

(b) Find the angle α, giving your answer in degrees accurate to 2 decimal places, in each of the following equations

$$2\sin 3\alpha = 1\cdot 4\tan 40^o$$

$$\tan\left(2\alpha - \frac{\pi}{2}\right) = 5\cdot 2$$

6 marks

(c) Consider the following vectors in $\mathbb{R}^3$.

$$P = (0, 8, -1) \quad , \quad Q = (5, 5, 0) \quad , \quad R = (8, 0, 1)$$

i Evaluate $P - 2Q + 3R$.

ii Determine $||2P - 4Q||$.

iii Evaluate $P.Q$, i.e., the *scalar product* of P with Q.

iv Determine the angle θ between P and Q.

8 marks

2. (a) Find a *unit vector* perpendicular to both of the vectors

$$P(2, 1, -1) \quad , \quad Q(1, -1, 2)$$

4 marks

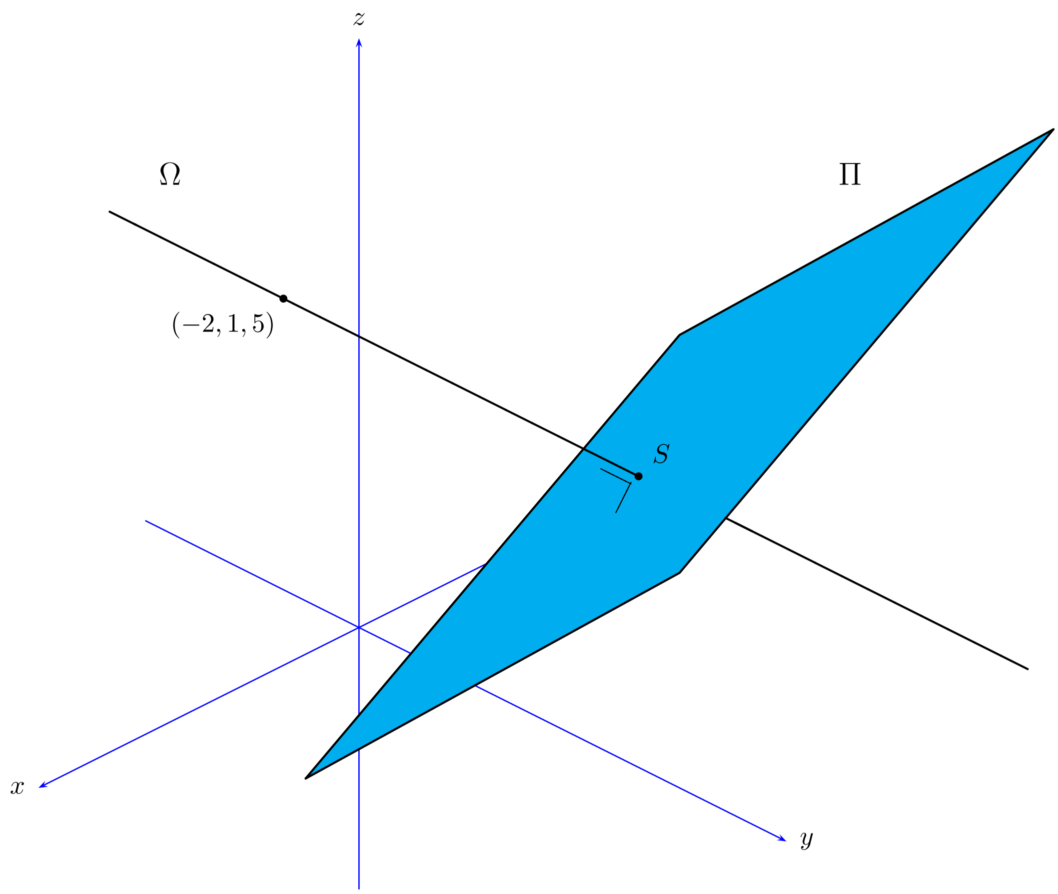
(b) Consider a line Ω through the point $(-2, 1, 5)$ that is perpendicular to the plane Π with equation $4x - 2y + 2z + 1 = 0$.

i Find the parametric equations of the line Ω.

ii Find the point S at which Ω meets Π.

iii Determine the perpendicular distance from the point $T(1, 2, 3)$ to the plane Π.

16 marks



3. Consider the following points in $\mathbb{R}^3$.

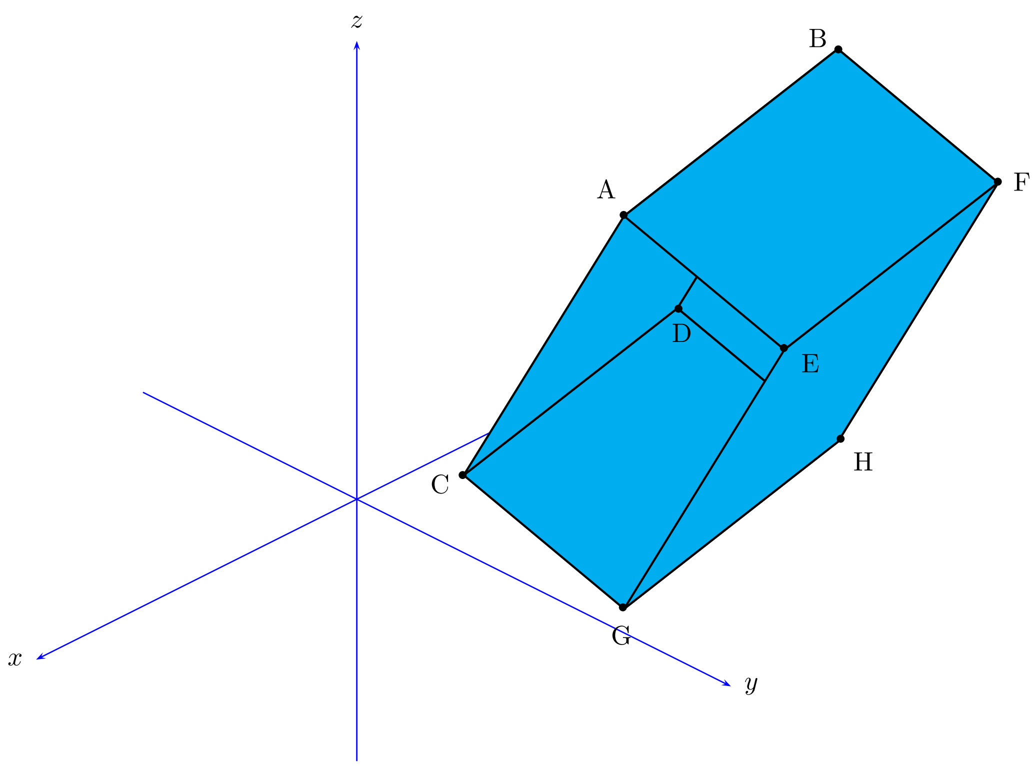
$$A(1,3,2) \quad , \quad B(3,7,3) \quad , \quad C(4,5,1) \quad , \quad D(6,9,2)$$
$$E(1,4,7) \quad , \quad F(3,8,8) \quad , \quad G(4,6,6) \quad , \quad H(6,10,7)$$

Note that

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{GH} = (2,4,1) \\ \overrightarrow{AC} &= \overrightarrow{BD} = \overrightarrow{EG} = \overrightarrow{FH} = (3,2,-1) \\ \overrightarrow{AE} &= \overrightarrow{BF} = \overrightarrow{CG} = \overrightarrow{DH} = (0,1,5) \end{aligned}$$

It follows that A, B, C, D, E, F, G, and H are the vertices of a *parallelepiped* in $\mathbb{R}^3$.

Let $\vec{u} = (2,4,1)$, $\vec{v} = (3,2,-1)$ and $\vec{w} = (0,1,5)$.



(a) Calculate the magnitude of the vectors $\overrightarrow{BG}$ and $\overrightarrow{BH}$ and the angle θ between these two vectors at the point B.

6 marks

(b) Determine the equation of the plane passing through the points A, B and F, expressing in the form $ax + by + cz + d = 0$ where a, b, c and d are constants.

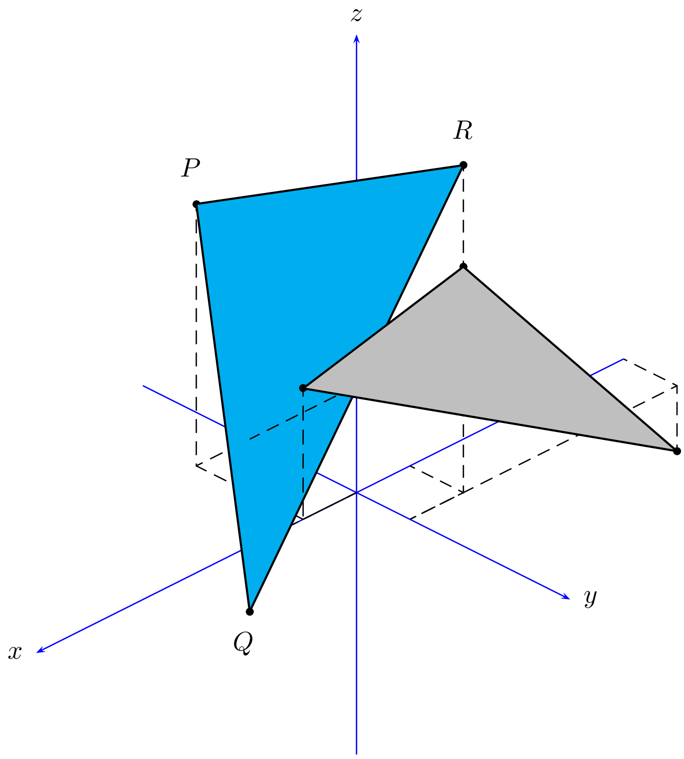
10 marks

(c) The volume V of the *parallelepiped* is defined as $V = \vec{u}.(\vec{v} \times \vec{w})$. Evaluate V.

4 marks

4. Consider the following points in $\mathbb{R}^3$.

$$P(1,-2,4) \quad , \quad Q(2,0,-1) \quad , \quad R(-1,1,5)$$



(a) Find the area of a triangle PQR.

8 marks

(b) Let $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that

$$T(x) = [A]x$$

denote a *matrix transformation* with *standard matrix* A. Rotate the triangle PQR *anti-clockwise* about the positive y-axis through an angle of 270^o where

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

6 marks

(c) Show that the *standard matrix* from part ii is *orthogonal*, i.e., $A^{-1} = A^t$.

Comment on the important property of orthogonal matrices when applied to the rotation of the triangle PQR from part ii.

6 marks

5. (a)

The plaque in this photo appears on Broome Bridge along the Royal Canal, Dublin. It commemorates the discovery of *quaternions* on 16th October, 1843.

i Name the 19^{th} century Irish mathematician who discovered quaternions.

ii Comment on the contribution quaternions make to computer graphics.

4 marks

(b) Given the quaternions

$$\begin{aligned} q_1 &= 1 + 4i + 6j + k \\ q_2 &= 2 - 4i + 4j - 4k \end{aligned}$$

Evaluate each of the following

i $4q_1 + q_2$,

ii $q_1 \times q_2$,

iii q_1^{-1}, the *inverse* of q_1.

10 marks

(c) Consider the point $r(-3, 4, 2)$ in $\mathbb{R}^3$.

Rotate this point 180^o *anti-clockwise* about the x-axis, (i.e., the vector $\vec{v} = (1, 0, 0)$) using a quaternion.

Note: Full workings must be shown for this question.

6 marks

6. (a) Let

$$A = \begin{pmatrix} 1 & 2 \\ -5 & 2 \\ -5 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 5 \\ -3 & 2 & 1 \end{pmatrix}$$

Determine each of the following

i $3A + B^t$

ii AB

6 marks

(b) Let

$$A = \begin{pmatrix} 3 & 5 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & 2 \end{pmatrix}$$

By a sequence of *row operations* find A^{-1}, the *inverse* of A.

Use this inverse to find the solution of the following system of linear equations.

$$\begin{aligned} 3x + 5y - z &= 3 \\ x + z &= 5 \\ -x - y + 2z &= 4 \end{aligned}$$

14 marks

7. (a) Find the parametric equation for the line that starts at the point $\vec{P_0} = (1, 3, 2)$ and points in the direction $\vec{v} = (0, 0 \cdot 5, 0 \cdot 5)$.

Calculate the position of three different points that lie on this line and draw a diagram that shows these three points on the line.

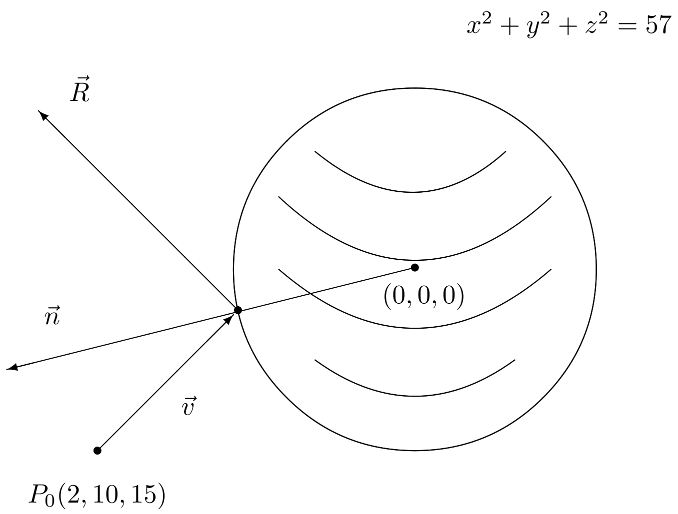
6 marks

(b) A sphere with a reflective surface is defined by the equation

$$x^2 + y^2 + z^2 = 57$$

A light source at $(2, 10, 15)$ with a very narrow beam is turned on and points in the direction $\vec{v} = (-1, -3, -2)$. Find where the centre of the beam of light will hit the sphere and the direction of the reflected light.

14 marks



USEFUL INFORMATION

VECTORS IN $\mathbb{R}^3$:

For any vector $\vec{u} = (u_1, u_2, u_3)$ in $\mathbb{R}^3$, we define the magnitude of $\vec{u}$ to be the non-negative real number

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Suppose that $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$ and that $\theta \in [0, \pi]$ represents the angle between them. We define the scalar product $\vec{u}.\vec{v}$ of $\vec{u}$ and $\vec{v}$ by $\vec{u}.\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$. Hence

$$\cos\theta = \frac{\vec{u}.\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

Alternatively, we can write $\vec{u}.\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$.

Theorem 1

Suppose that $\vec{u}, \vec{v} \in \mathbb{R}^3$. Then the parallelogram with $\vec{u}$ and $\vec{v}$ as two of its sides has area $\|\vec{u} \times \vec{v}\|$.

Theorem 2

The perpendicular distance D of a point $P = (x_0, y_0, z_0)$ from the plane $ax+by+cz+d = 0$ is given by

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

LINES AND PLANES IN $\mathbb{R}^3$:

Let Ω be a **line** in $\mathbb{R}^3$.

i Find a point $P_0 = (x_0, y_0, z_0)$ which is on Ω.

ii Find a vector $\vec{v} = (v_1, v_2, v_3)$ which is parallel to Ω.

Then $t\vec{v}$ is also parallel to Ω and the line is the set of all points $P = (x, y, z)$ for which the vector

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

is equal to $t\vec{v}$, or

$$\overrightarrow{P_0P} = t\vec{v}$$

for some $t \in \mathbb{R}$.

Let Π be a **plane** in $\mathbb{R}^3$.

i Find the co-ordinates of a point $P_0 = (x_0, y_0, z_0)$ which is in the plane.

ii Find a vector $\vec{n} = (a, b, c)$ perpendicular to the plane.

Then the plane Π consists of those points $P = (x, y, z)$ for which the vector

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

is orthogonal to $\vec{n}$, or

$$\overrightarrow{P_0P}.\vec{n} = 0$$

QUATERNIONS:

From the formula defining multiplication of quaternions we have that

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k \quad , \quad jk = -kj = i \quad , \quad ki = -ik = j$$

If follows directly from these identities that $ijk = -1$. The operation of multiplication on the set $\mathbb{H}$ of quaternions is **not** commutative.

QUATERNIONS AND ROTATIONS:

The following quaternion

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(li + mj + nk)$$

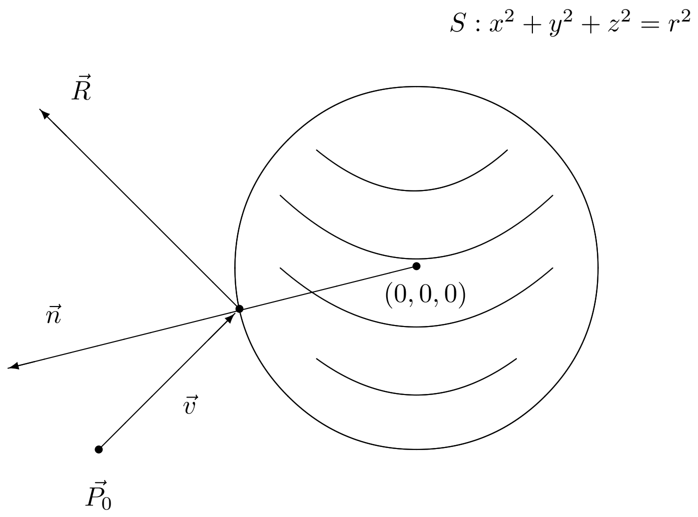
will rotate a point $r(x, y, z)$ through θ *anti-clockwise* about the unit vector $\vec{n} = (l, m, n)$. The rotation will be achieved by $\mathbf{Rotation} = \mathbf{q}.\mathbf{r}.\mathbf{q}^{-1}$ where $r(x, y, z) = xi + yj + zk$.

Note also that for a quaternion $q = w + xi + yj + zk$

$$q^{-1} = \frac{1}{|q|^2}\bar{q}$$

the modulus is given as $|q|^2 = w^2 + x^2 + y^2 + z^2$ and the conjugate is given as $\bar{q} = w - xi - yj - zk$

RAY TRACING :



The equation of the beam of light (starting at P_0 and in the direction $\vec{v}$) is given as

$$\vec{P} = \vec{P}_0 + t.\vec{v}$$

The vector $\vec{L}$ is a vector **pointing towards** the light source, i.e., $\vec{L} = -\vec{v}$ and the vector $\vec{R}$ represents the direction of the reflected light. If the normal to the surface is $\vec{n}$ and ensuring the vectors $\vec{L}$ and $\vec{n}$ are **unit vectors**, then the reflected beam is represented as

$$\vec{R} \ = \ 2(\vec{n}.\vec{L})\vec{n} - \vec{L}$$