INSTRUCTIONS

Full marks will be awarded for the correct solutions to **ANY FIVE QUESTIONS**. This paper will be marked out of a TOTAL MAXIMUM MARK OF 100. Credit will be given for clearly presented solutions. **MATHEMATICAL TABLES**, if required, are available from the invigilator. Take note of the **USEFUL INFORMATION** presented with this examination paper.

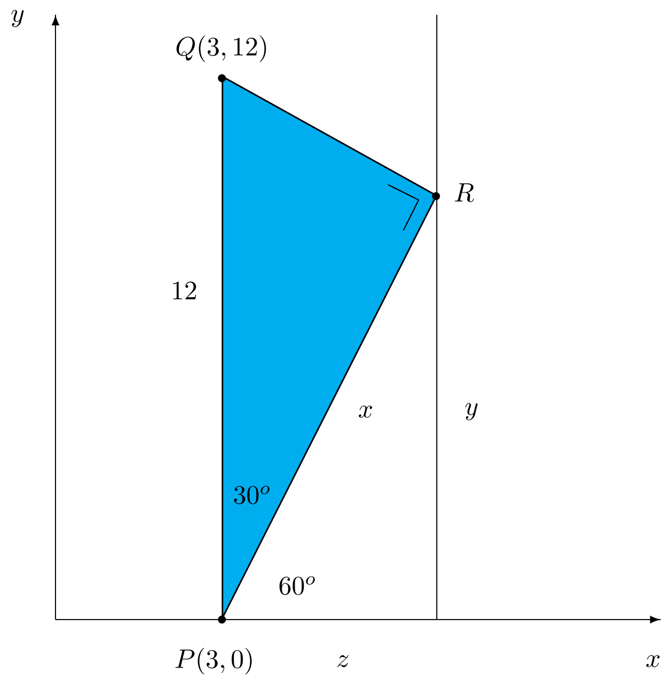
CW_ KCCGD_ B

BSc (Hons) in Computer Games Development

YEAR 1

SUMMER, 2019

1. (a)



$$\cos 30^o = x/12$$

$$x = 12\cos 30^o = 6\sqrt{3}$$

$$\sin 60^o = y/6\sqrt{3}$$

$$y = 6\sqrt{3}\sin 60^o = 9$$

$$\cos 60^o = z/6\sqrt{3}$$

$$z = 6\sqrt{3}\cos 60^o = 3\sqrt{3}$$

$$R = (3 + 3\sqrt{3}, 9)$$

$$R = (8 \cdot 2, 9)$$

Now $P(8,5,-2), Q(1,4,-1), R(4,5,-10)$.

i

$$3P - Q + 3R = 3(8,5,-2) - (1,4,-1) + 3(4,5,-10) = (35,26,-35)$$

ii

$$\begin{aligned} P + 5Q &= (8,5,-2) + 5(1,4,-1) = (13,25,-7) \\ ||P + 5Q|| &= \sqrt{169 + 625 + 49} = \sqrt{843} = 29 \cdot 03 \end{aligned}$$

iii

$$P.Q = 8.1 + 5.4 + (-2).(-1) = 30$$

iv

$$P \times Q = det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 8 & 5 & -2 \\ 1 & 4 & -1 \end{vmatrix} = (3,6,27)$$

v

$$\begin{aligned} \vec{PQ} &= \vec{Q} - \vec{P} = (1,4,-1) - (8,5,-2) = (-7,-1,1) \\ \vec{PR} &= \vec{R} - \vec{P} = (4,5,-10) - (8,5,-2) = (-4,0,-8) \\ \vec{RQ} &= \vec{Q} - \vec{R} = (1,4,-1) - (4,5,-10) = (-3,-1,9) \end{aligned}$$

$$\vec{PQ} \times \vec{PR} = det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -7 & -1 & 1 \\ -4 & 0 & -8 \end{vmatrix} = (8,-60,-4)$$

$$\vec{RQ}.(\vec{PQ} \times \vec{PR}) = -3(8) - 1(-60) + 9(-4) = -24 + 60 - 36 = 0$$

The vectors defining this triangle $\triangle PQR$ are *coplanar*.

vi To determine the angle α

$$\cos\alpha = \frac{\vec{PQ}.\vec{PR}}{\|\vec{PQ}\|\|\vec{PR}\|} = \frac{20}{\sqrt{51}.\sqrt{80}} = 0 \cdot 3131 \quad , \quad \alpha = 71 \cdot 8^o$$

To determine the angle β

$$\cos\beta = \frac{\vec{RP}.\vec{RQ}}{\|\vec{RP}\|\|\vec{RQ}\|} = \frac{60}{\sqrt{93}.\sqrt{91}} = 0 \cdot 6522 \quad , \quad \beta = 49 \cdot 3^o$$

Finally $\alpha + \beta + \gamma = 180^o$, hence $\gamma = 58 \cdot 9^o$.

vii Hence the area of the triangle $\triangle PQR$ is

$$\frac{1}{2}\|\vec{PQ} \times \vec{PR}\|$$

where $\vec{PQ} \times \vec{PR} = (8, -60, -4)$. Now

$$\frac{1}{2}\|\vec{PQ} \times \vec{PR}\| = \frac{1}{2}\sqrt{64 + 3600 + 16} = \frac{1}{2}\sqrt{3680}$$

The area of the triangle $\triangle PQR$ is $2\sqrt{230}$ *sq. units*.

2. (a) The line Ω goes through the points $P_0(-2, 1, 5)$ and is *perpendicular* to the plane Π with equation $4x - 2y + 2z + 1 = 0$, hence $\vec{v} = (4, -2, 2)$. Now

$$\begin{aligned} \overrightarrow{P_0P} &= \vec{P} - \vec{P_0} \\ &= (x, y, z) - (-2, 1, 5) \\ &= (x + 2, y - 1, z - 5) \end{aligned}$$

Now $\overrightarrow{P_0P} = t\vec{v}$, hence

$$\begin{aligned} (x + 2, y - 1, z - 5) &= t(4, -2, 2) \\ &= (4t, -2t, 2t) \end{aligned}$$

for some $t \in \mathbb{R}$. Expanding gives the parametric equations of the line Ω.

$$\begin{aligned} x &= -2 + 4t \\ y &= 1 - 2t \\ z &= 5 + 2t \end{aligned}$$

(b) Now to calculate the point of intersection of the line Ω and the plane $\Pi : \; 4x - 2y + 2z + 1 = 0$.

$$\begin{aligned} 4(-2 + 4t) - 2(1 - 2t) + 2(5 + 2t) + 1 &= 0 \\ -8 + 16t - 2 + 4t + 10 + 4t + 1 &= 0 \\ 24t &= -1 \\ t &= -\frac{1}{24} \end{aligned}$$

Hence, the point of intersection is

$$\begin{aligned} x &= -2 + 4(-\frac{1}{24}) = -\frac{13}{6} \\ y &= 1 - 2(-\frac{1}{24}) = \frac{13}{12} \\ z &= 5 + 2(-\frac{1}{24}) = \frac{59}{12} \end{aligned}$$

Therefore, $P = (-\frac{13}{6}, \frac{13}{12}, \frac{59}{12})$.

(c) Finally, the perpendicular distance D of the point $A(1, 2, 3)$ from the plane Π $4x - 2y + 2z + 1 = 0$ is given as

$$D = \frac{|4(1) - 2(2) + 2(3) + 1|}{\sqrt{4^2 + (-2)^2 + 2^2}} = \frac{7}{\sqrt{24}}$$

3. For the given planes Π_1 and Π_2 we have $\vec{n}_1 = (2, 1, -3)$ and $\vec{n}_2 = (1, -1, 3)$. Now

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ 1 & -1 & 3 \end{pmatrix} = (0, -9, -3)$$

Given that the plane whose equation we are seeking is perpendicular to the line of intersection of the planes Π_1 and Π_2, we say that the normal vector $\vec{n}_\pi$, that is required to give us this plane, is equal the vector $\vec{v}$. Therefore $\vec{n}_\pi = (0, -9, -3)$ and hence

$$\Pi : \; -9y - 3z + d = 0$$

But $P_0 = (1, 1, -8) \in \Pi$, therefore

$$\begin{aligned} -9(1) - 3(-8) + d &= 0 \\ -9 + 24 + d &= 0 \\ d &= -15 \end{aligned}$$

Finally

$$\Pi : \; -9y - 3z - 15 = 0$$

$$\Pi : \; 3y + z + 5 = 0$$

Now, our three planes are as follows:

$$\begin{aligned} 2x + y - 3z &= 1 \\ x - y + 3z &= 0 \\ 3y + z &= -5 \end{aligned}$$

To find the point of intersection we will write the augmented matrix that represents this system of linear equations and use a sequence of row operations to bring the matrix to row-echelon form. The augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 1 & -3 & 1 \\ 1 & -1 & 3 & 0 \\ 0 & 3 & 1 & -5 \end{array}\right)$$

We now apply row operations

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 2 & 1 & -3 & 1 \\ 0 & 3 & 1 & -5 \end{array}\right)$$

$$\xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 3 & -9 & 1 \\ 0 & 3 & 1 & -5 \end{array}\right)$$

$$\xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 3 & -9 & 1 \\ 0 & 0 & 10 & -6 \end{array}\right)$$

Back substitution yields

$$z = \frac{-3}{5}\ ,\ y = \frac{-22}{15}\ ,\ x = \frac{5}{15}$$

Finally, the point of intersection is

$$P = \left(\frac{5}{15}, \frac{-22}{15}, \frac{-3}{5}\right)$$

4. (a)

$$A = \begin{pmatrix} 1 & 0 \\ 5 & 8 \\ -5 & 7 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 4 & 5 \\ -3 & 2 & 0 \end{pmatrix}$$

i

$$\begin{aligned} 3A + B^t &= 3\begin{pmatrix} 1 & 0 \\ 5 & 8 \\ -5 & 7 \end{pmatrix} + \begin{pmatrix} 4 & 4 & 5 \\ -3 & 2 & 0 \end{pmatrix}^t \\ &= \begin{pmatrix} 3 & 0 \\ 15 & 24 \\ -15 & 21 \end{pmatrix} + \begin{pmatrix} 4 & -3 \\ 4 & 2 \\ 5 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 7 & -3 \\ 19 & 26 \\ -10 & 21 \end{pmatrix} \end{aligned}$$

ii

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 0 \\ 5 & 8 \\ -5 & 7 \end{pmatrix} \cdot \begin{pmatrix} 4 & 4 & 5 \\ -3 & 2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 4 & 5 \\ -4 & 36 & 25 \\ -41 & -6 & -25 \end{pmatrix} \end{aligned}$$

(b) We form the following matrix

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

Now

$$\frac{R_2 - 2R_1}{R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\underline{R_3 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \\ 0 & 2 & -1 & -2 & 1 & 0 \end{array}\right)$$

$$\underline{R_3 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -4 & 1 & 2 \end{array}\right)$$

$$\underline{R_2 - R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & -4 & 1 & 2 \end{array}\right)$$

$$\underline{R_2 \times (-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 1 \\ 0 & 0 & 1 & -4 & 1 & 2 \end{array}\right)$$

Therefore, we have

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 1 \\ -4 & 1 & 2 \end{pmatrix}$$

ii A system of 3 linear equations in 3 unknowns may be represented as $Ax = B$. We can now use the inverse A^{-1} to solve the system using matrix multiplication as follows

$$\begin{aligned} A^{-1}Ax &= A^{-1}B \\ \Rightarrow \quad Ix &= A^{-1}B \\ \therefore \quad x &= A^{-1}B \end{aligned}$$

hence

$$x = A^{-1}B = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 1 \\ -4 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \\ 12 \end{pmatrix}$$

i.e. $x = -1$, $y = 8$ and $z = 12$.

5. (a)

$$P(1,2,3) \quad , \quad Q(3,4,0) \quad , \quad R(2,0,1)$$

Firstly

$$\begin{aligned} \vec{PQ} &= \vec{Q} - \vec{P} = (3,4,0) - (1,2,3) = (2,2,-3) \\ \vec{PR} &= \vec{R} - \vec{P} = (2,0,1) - (1,2,3) = (1,-2,-2) \\ \vec{RQ} &= \vec{Q} - \vec{R} = (3,4,0) - (2,0,1) = (1,4,-1) \end{aligned}$$

$$\vec{PQ} \times \vec{PR} = det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & -3 \\ 1 & -2 & -2 \end{vmatrix} = (-10, 1, -6)$$

The area of the triangle $\triangle PQR$ is

$$\frac{1}{2}\|\vec{PQ} \times \vec{PR}\|$$

where $\vec{PQ} \times \vec{PR} = (-10, 1, -6)$. Now

$$\frac{1}{2}\|\vec{PQ} \times \vec{PR}\| = \frac{1}{2}\sqrt{100+1+36} = \frac{1}{2}\sqrt{137}$$

The area of the triangle $\triangle PQR$ is $5 \cdot 85$ *sq. units*

(b)

$$T[P] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$$

$$T[Q] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$$

$$T[R] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

(c) To show that A is orthogonal, note that a square matrix A is an *orthogonal* matrix if $A^{-1} = A^t$. Pre-multiplying both sides by A yield

$$A.A^{-1} = I = A.A^t$$

Now

$$A.A^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

When an orthogonal matrix is used to rotate vectors, it will keep the lengths of the vectors preserved as will the angle between the vectors. So any orthogonal matrix may represent a rotation.

6. (a) The equation of the beam of light is

$$\vec{P} = \vec{P_0} + t.\vec{v}$$

$$\begin{aligned} \vec{P} &= (7,2,2) + t(-1,-1,-1) \\ &= (7-t, 2-t, 2-t) \end{aligned}$$

This ray will intersect the sphere when

$$(7-t)^2 + (2-t)^2 + (2-t)^2 = 25$$

$$3t^2 - 22t + 32 = 0$$

which has solution $t = 5{\cdot}33$ and $t = 2$, using the quadratic formula. The parametric equations for the ray is $\vec{P} = (7,2,2) + t(-1,-1,-1)$, hence the two points of intersection are

$$\begin{aligned} \vec{P}(5\cdot 33) &= (7,2,2) + 5\cdot 33(-1,-1,-1) = (1\cdot 66, -3\cdot 33, -3\cdot 33) \\ \vec{P}(2) &= (7,2,2) + 2(-1,-1,-1) = (5,0,0) \end{aligned}$$

Since the ray starts at the point $(7,2,2)$ it will hit the sphere at the point $(5,0,0)$ first.

(b) To find the reflected ray from the point $(5,0,0)$ we use the equation $\vec{R} = 2(\vec{n}.\vec{L})\vec{n} - \vec{L}$ where the unit direction vector $-\vec{v}$, **towards the light** is

$$\vec{L} = \frac{(1,1,1)}{||\vec{v}||} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

and the unit normal vector is

$$\vec{n} = \frac{(5,0,0)}{5} = (1,0,0)$$

The reflected beam is in the direction

$$\vec{R} = 2(\vec{n}.\vec{L})\vec{n} - \vec{L}$$

$$\vec{R} = \frac{2}{\sqrt{3}}(1,0,0) - \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\vec{R} = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

7. (a) Given the *quaternions*

$$\begin{aligned} q_1 &= 4+i+4j+k \\ q_2 &= 1+6i+6j-2k \end{aligned}$$

i

$$\begin{aligned} q_1+6q_2 &= (4+i+4j+k)+6(1+6i+6j-2k) \\ &= (4+6)+(1+36)i+(4+36)j+(1-12)k \\ &= 10+37i+40j-11k \end{aligned}$$

ii

$$\begin{aligned} q_1 \times q_2 &= (4+i+4j+k)\times(1+6i+6j-2k) \\ &= (4+24i+24j-8k)+(i+6i^2+6ij-2ik) \\ &\quad +(4j+24ji+24j^2-8jk)+(k+6ki+6kj-2k^2) \\ &= (4-6-24+2)+24i+24j-8k+i+6k+2j+4j-24k-8i+k+6j-6i \\ &= -24+11i+36j-25k \end{aligned}$$

iii Now $\bar{q_1} = 4-i-4j-k$ and $|q_1|^2 = 4^2+1^2+4^2+1^2 = 34$. Hence

$$q_1^{-1} = \frac{1}{34}(4-i-4j-k)$$

(b) Rotate the point $r(4,2,8)$ 180^o *anti-clockwise* about the z-axis, (i.e., the vector $\vec{v} = (0,0,1)$) using a quaternion.

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(li+mj+nk) = \cos\frac{180}{2} + \sin\frac{180}{2}(0i+0j+k)$$

Hence $q = k$. Also $q^{-1} = -k$ and $r = 4i+2j+8k$. Now

$$\begin{aligned} Rotation &= q.r.q^{-1} \\ &= k.(4i+2j+8k).-k \\ &= (4ki+2kj+8k^2).-k \\ &= (4j-2i-8).-k \\ &= -4jk+2ik+8k \\ Rotation &= -4i-2j+8k \end{aligned}$$

Hence

$$r(4,2,8) \longrightarrow r'(-4,-2,8)$$