1. (a) Given the *quaternions*

$$\begin{aligned} q_1 &= 1+4i+2j-k \\ q_2 &= 2-4i+5j+2k \end{aligned}$$

i

$$\begin{aligned} 3q_1+q_2 &= 3(1+4i+2j-k)+(2-4i+5j+2k) \\ &= (3+2)+(12-4)i+(6+5)j+(-3+2)k \\ &= 5+8i+11j-k \end{aligned}$$

5 marks

ii

$$\begin{aligned} q_1 \times q_2 &= (1+4i+2j-k)\times(2-4i+5j+2k) \\ &= (2-4i+5j+2k)+(8i-16i^2+20ij+8ik) \\ &\qquad +(4j-8ji+10j^2+4jk)+(-2k+4ki-5kj-2k^2) \\ &= (2+16-10+2)-4i+5j+2k+8i+20k-8j+4j+8k+4i-2+4j+5i \\ &= 10+13i+5j+28k \end{aligned}$$

15 marks

(b) Rotate the point $r(2,2,2)$ 180^o *anti-clockwise* about the y-axis, (i.e., the vector $\vec{v} = (0,1,0)$) using a quaternion.

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(li+mj+nk) = \cos\frac{180}{2} + \sin\frac{180}{2}(0i+j+0k)$$

Hence $q = j$. Also $q^{-1} = -j$ and $r = 2i+2j+2k$. Now

$$\begin{aligned} Rotation &= q.r.q^{-1} \\ &= j.(2i+2j+2k).-j \\ &= (2ji+2j^2+2jk).-j \\ &= (-2+2i-2k).-j \\ &= 2j-2ij+2kj \\ Rotation &= -2i+2j-2k \end{aligned}$$

Hence

$$r(2,2,2) \longrightarrow r'(-2,2,-2)$$

20 marks

2. i The line Ω goes through the points $P_0(-2,1,5)$ and is *perpendicular* to the plane Π with equation $4x-2y+2z+1=0$, hence $\vec{v}=(4,-2,2)$. Now

$$\begin{aligned} \overrightarrow{P_0P} &= \vec{P}-\vec{P_0} \\ &= (x,y,z)-(-2,1,5) \\ &= (x+2,y-1,z-5) \end{aligned}$$

Now $\overrightarrow{P_0P}=t\vec{v}$, hence

$$\begin{aligned} (x+2,y-1,z-5) &= t(4,-2,2) \\ &= (4t,-2t,2t) \end{aligned}$$

for some $t\in\mathbb{R}$. Expanding gives the parametric equations of the line Ω.

$$\begin{aligned} x &= -2+4t \\ y &= 1-2t \\ z &= 5+2t \end{aligned}$$

25 marks

ii Now to calculate the point of intersection of the line Ω and the plane $\Pi:\ 4x-2y+2z+1=0$.

$$\begin{aligned} 4(-2+4t)-2(1-2t)+2(5+2t)+1 &= 0 \\ -8+16t-2+4t+10+4t+1 &= 0 \\ 24t &= -1 \\ t &= -\frac{1}{24} \end{aligned}$$

Hence, the point of intersection is

$$\begin{aligned} x &= -2+4(-\frac{1}{24}) = -\frac{13}{6} \\ y &= 1-2(-\frac{1}{24}) = \frac{13}{12} \\ z &= 5+2(-\frac{1}{24}) = \frac{59}{12} \end{aligned}$$

Therefore, $P=(-\frac{13}{6}\ ,\frac{13}{12},\frac{59}{12})$.

20 marks

iii Finally, the perpendicular distance D of the point $T(1,2,3)$ from the plane Π $4x-2y+2z+1=0$ is given as

$$D=\frac{|4(1)-2(2)+2(3)+1|}{\sqrt{4^2+(-2)^2+2^2}}=\frac{7}{\sqrt{24}}$$

15 marks